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The Queen of the Sciences:
A History of Mathematics

Taught by: David M. Bressoud
Macalester College

Parts 1 & 2

Course Guidebook

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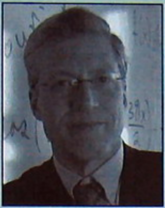

Professor David M. Bressoud is DeWitt Wallace Professor of Mathematics at Macalester College. He earned his bachelor's degree from Swarthmore College and his Ph.D. in Mathematics from Temple University. Professor Bressoud received Macalester College's Thomas Jefferson Award and the Award for Distinguished College or University Teaching of Mathematics from the Allegheny Mountain Section of the Mathematical Association of America. He was elected president of the Mathematical Association of America for 2009–2011.

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Parts I & II

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David Bressoud earned a Bachelor's degree in Mathematics from Swarthmore College in 1971; joined the Peace Corps, where he taught mathematics and science at the Clare Hall School in Antigua, West Indies; and then earned a Ph.D. in Mathematics at Temple University in 1977. He taught for 17 years at the University Park campus of The Pennsylvania State University before moving to Macalester College in 1994, where he chaired the Department of Mathematics and Computer Science from 1995 until 2001. He has received a Sloan Foundation Fellowship and a Fulbright award and has held visiting positions at the Institute for Advanced Study, the University of Wisconsin–Madison, the University of Minnesota, Université Louis Pasteur (Strasbourg, France), and State College Area High School. His research interests lie in number theory, combinatorics, and analysis, and he has published more than 50 research papers.

Professor Bressoud's interest in the history of mathematics was sparked by a series of lectures by Subrahmanyan Chandrasekhar on Newton's *Principia* and by his own research into the mathematics of the self-taught genius Srinivasa Ramanujan. He has since taught courses on the history of mathematics in South Asia and on Newton's *Principia* and the scientific revolution.

In addition to two textbooks on number theory, *Factorization and Primality Testing* (1989) and *A Course in Computational Number Theory* (2000; coauthored with Stan Wagon), Professor Bressoud has written four textbooks that draw on the history of mathematics to motivate its study: *Second Year Calculus from Celestial Mechanics to Special Relativity* (1991); *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture* (1999); *A Radical Approach to Real Analysis* (2nd edition, 2007); and *A Radical Approach to Lebesgue's Theory of Integration* (2008). In 1994 he won the Award for Distinguished College or University Teaching of Mathematics from the Allegheny Mountain Section of the Mathematical Association of America (MAA). He has also received the MAA's Beckenbach Book Prize for *Proofs and Confirmations* (2000) and Macalester College's Thomas Jefferson Award for his "personal influence, teaching, writing and scholarship" (2005). Professor Bressoud has been elected president of the Mathematical Association of America for 2009–2010.

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Scope:

These lectures describe the historical development of mathematics from the earliest records of Mesopotamia and Egypt up to the problems and challenges of mathematics today. The goal is to convey the nature and power of this discipline while exploring the lives of many of the people who have shaped it. Although our focus is on the great figures, we will also study the men—and occasional women—who laid the groundwork for their discoveries, acted as their collaborators, and built on their insights.

The history of mathematics covers 4000 years and many different cultures and civilizations. These lectures have been structured so that developments of the 17th century in Europe form the pivot. That is the critical time and place in which developments in algebra, geometry, astronomy, mechanics, and the mathematics of motion reach the level of sophistication needed to weave them into a structure that would enable the rapid development of mathematics in the succeeding centuries. The 17th century witnessed the emergence of analytic geometry and calculus as well as important foundational advances in geometry, probability, and number theory.

Lectures Ten through Fourteen are devoted to the mathematical advances of the 17th century, focusing on seven great people: Galileo, a modern figure forever constrained by Aristotelian assumptions; Napier, the Scotsman who would use logical reasoning to simplify complex calculations; Fermat, the parliamentary lawyer who would establish the foundations of calculus, set number theory on a new course, and discover a problem that would challenge the greatest minds of the 20th century; Newton, whom Keynes described as the "last of the magicians"; Leibniz, the court librarian who would fight with Newton for bragging rights to the creation of calculus; and the Bernoullis, two incredibly gifted Swiss brothers who would become Leibniz's collaborators.

The lectures leading up to the 17th century explore the question of how mathematics arrived at this golden moment. We begin by considering the definition of mathematics and the question of why it has been so unreasonably effective at explaining the physical world. Our exploration of the historical record begins with Babylonian and Egyptian mathematics of 2000 B.C. Already, this is very sophisticated mathematics that uses the Pythagorean theorem, solves quadratic equations, and calculates square roots to high precision. We will spend three lectures on Greek and

Hellenistic mathematics. The first will follow its development through Thales, Pythagoras, Theaetetus, and Eudoxus up to the crowning achievement of Euclid's *Elements*. In the next lecture, we will focus on three great Hellenistic figures: Archimedes, Apollonius, and Diophantus.

We will conclude our study of this period with an exploration of the early development of trigonometry and the contributions of Hipparchus and Ptolemy. We continue with the Indian, Chinese, and Islamic mathematics that continued this chain of development toward the 17th century, including computational advances from China, trigonometry from India, and algebra from the Islamic caliphates. The final preparation would come from the European algebraists, focusing particularly on Tartaglia and Cardano, the feuding Italian algebraists of the 16th century.

Following the 17th century, this survey of mathematical history is necessarily much sketchier. The creative power and complexity of the mathematics of the past 300 years is so great that we can no more than sample a few of its results. I have chosen the people and mathematical developments that I consider most interesting, among them Euler, the man who dominated 18th-century mathematics even after he had gone blind; and Gauss, the "prince of mathematics," whose reluctance to publish until a result was perfect would frustrate many of his contemporaries. Several of these mathematicians of the past 300 years died tragically young: Galois at 20, Abel at 26, Ramanujan at 32, and Riemann at 39. Several, such as Jacob Jacobi, James Joseph Sylvester, Sophie Germain, and Sofya (Sonya) Kovalevskaya, had to fight prejudice against Jews or women.

In these last lectures, we will investigate what happened to calculus in the 19th century as it came to be known as *analysis*. We will look at developments in geometry and algebra as both began to focus on invariants, the fundamental characteristics that do not change. We also will explore the development of elliptic and modular functions that arose in extending the notion of trigonometric functions and that would prove crucial to modern physics. We will follow some of the recent developments in number theory, including the proof of Fermat's last theorem. We will examine the role of mathematics in predicting physical reality, as exhibited in the work of Maxwell and Einstein. And we will conclude with a selective survey of current problems and areas of study in mathematics.

Lecture One
What Is Mathematics?

Scope: This lecture explores the nature of mathematics, a subject that arose from the abstraction of patterns observed in the world around us and developed as those abstractions were codified and pushed beyond practical applications. One of the mysteries of mathematics is its unreasonable effectiveness: Why it is that abstractions that arose in one context can lead to unexpected insights when applied to a totally different situation? We will explore how the history of mathematics can help us understand the true nature and power of this subject. The lecture concludes by outlining the path this course will take over 4000 years in the history of mathematics.

Outline

I. Mathematics is about ideas, and one of the most effective ways of understanding mathematical ideas is to see how they developed. Where did they come from? What were people trying to understand as they looked at these ideas? How did the ideas arise from different civilizations and cultures?

  A. In exploring mathematical ideas, it's also important to know where difficulties arose in their development. As we'll see, great mathematicians have struggled with some of the ideas we'll explore; thus, it's no surprise that students may struggle with them as well.

  B. The history of mathematics is also full of great stories and interesting people. In this course, we'll meet such figures as Leonhard Euler, a great mathematician who ultimately went blind; Evariste Galois, who solved one of the most fundamental problems in mathematics at the age of 17; and the self-taught Indian mathematician Srinivasa Ramanujan.

II. Before we launch into the history of mathematics, we should define the field.

  A. The essence of mathematics is the abstraction of pattern. Mathematicians pick out patterns from the world around us and then abstract those patterns in order to manipulate them and tell us more about the world.

B. Some of the simplest examples of this abstraction of pattern come from numbers. For example, we consider number simply as a qualifier: 5 people, 5 stars. From that, we can abstract the number 5. That abstraction is useful as we begin to combine objects; it becomes something that goes beyond the objects themselves.

C. The same thing happens in geometry. The basic geometric objects, such as lines and circles, express spatial relationships. We might think of a tree at 1 point, a house at another point, and a person standing in between them. We consider these 3 points and abstract them into a triangle.

D. What is the reality behind these abstractions? Is 5 something that actually exists? Does a pure Platonic circle really exist? The adjective *Platonic* reminds us of Plato's view that these ideal abstractions do exist, that there is a reality beyond ourselves that we tap into as we make these abstractions.

E. This leads to an even more fundamental question: Was mathematics discovered or created? I believe that as we develop these abstractions of patterns, we are trying to explain a deeper reality that does exist. Most mathematicians and scientists realize that we can never truly describe this reality, but mathematics creates a language, a set of symbols, that enables us to work with aspects of this deeper reality.

F. Mathematicians look for points of similarity between the patterns they discover. These points of similarity, in turn, suggest ways in which the pattern might be extended. This kind of exploration, the search for new knowledge of the world, is what makes mathematics exciting.

III. The patterns of mathematics come from many sources, as we'll see throughout this series of lectures.

A. One of the most important sources is commerce and civil administration.

1. This source gives us basic arithmetic—the fundamental operations of addition, subtraction, multiplication, and division—and simple fractions, as well as the whole numbers, or integers.

2. Commerce and civil administration also give rise to rates. We may need to know, for example, how long it will take to build a wall; what is the rate at which this job can be accomplished? We may need to know the yield for a given plot of land; how much is produced on each acre?

3. Commerce and civil administration are also the origins of much work in algebra, which in its earliest forms is closely connected to geometry.

B. Another source of the patterns in mathematics is navigation and surveying.

1. The need to make land measurements gave rise to geometry, which literally means "measuring the earth." Measurements of distances, areas, and volumes were needed.

2. As we'll see, geometry motivated work in algebra and number theory. Number theory relates to understanding the structure of integers. How can the integer 20 be represented? Can we write it as the sum of two squares? How can we decompose 20 into a product of primes?

C. A third source of the patterns of mathematics is astronomy or astrology.

1. The ancients made no clear distinction between these two fields. They studied the heavens to try to understand what was likely to happen on earth.

2. Some of the greatest astronomers, including Johannes Kepler, were also astrologers. Kepler's work in both astronomy and astrology would lay the foundations for much of the development of calculus.

3. This lack of distinction between astronomers and astrologers carried over to mathematicians. The emperor Tiberius is said to have banished all "mathematicians" from Rome. In fact he banished the astrologers, who were predicting his downfall.

4. Some of this confusion stems from the fact that important advances in mathematics came directly out of astronomy. By looking at the heavens, mathematicians were able to pick out patterns in much purer form than they could in the world around them.

5. We'll also see mathematics arising from other physical phenomena such as optics, electricity, and magnetism, and even the study of subatomic particles. Much mathematics has been created from the search for explanations of physical phenomena.

D. Yet another source of mathematical patterns is art and architecture.
   1. In later lectures, we'll see how the ideas of symmetry would come to play a fundamental role in the mathematics developed in the 19th and 20th centuries.
   2. The Lion Court at Alhambra in Granada is a wonderful example of symmetry in art; through this kind of symmetry, many mathematicians have seen the kinds of patterns they could apply to other problems, both in geometry and algebra.

IV. Let's look further at number and distance.
   A. At some point in antiquity, someone decided to apply the idea of number to distance, but there is no natural unit for measuring distances. Feet or meters, for example, are human constructs. Even once we decide on a unit, that unit is not necessarily appropriate for the distance we're trying to measure.
   B. Let's say that we're going to measure a distance using a stick. Almost inevitably, as we get close to the endpoint, we find that the distance left is less than the length of the stick. One way to solve this problem is to mark the stick off into smaller intervals.
   C. Again, however, the smaller pieces may not measure the remaining distance exactly. We can subdivide the smaller pieces, but we begin to realize that this process will be never ending. Does it even make sense to talk about the distance between two points as a number if we can never express that number exactly? Babylonian and Greek mathematicians wrestled with this question.
   D. Trying to apply number to time introduces other problems. Unlike distance, for which there is no natural unit, for time, there are too many units. Time involves the day, the lunar month, and the solar year, each of which is incommensurable with the other. We cannot measure a lunar month in a precise number of days. We cannot measure a solar year in an exact number of lunar months or an exact number of days. Whichever of these units we use, we will have a little bit left over.
   E. Further, these different units are traveling in different cycles; this would present fascinating problems for mathematicians, such as how to determine when all these cycles would line up or when different planets would be in alignment.

V. Two famous papers will be useful in our exploration of the nature of mathematics.

   A. The first of these was published in 1960 by the physicist Eugene Wigner, who taught at Princeton. Wigner's paper is called "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."
      1. Wigner noted that physicists seek to understand patterns in the world around them; they then look to mathematics to see if they can find similar patterns. Can the patterns in mathematics be applied to physical phenomena?
      2. Wigner remarked that in this comparison of patterns in the natural world and patterns from mathematics, the mathematical pattern often fits the physical phenomenon with amazing accuracy. Further, the mathematical pattern frequently offers excess content. It is possible to use the mathematical model to see beyond the original phenomenon.
      3. In other words, not only does the mathematical model enable us to make predictions about what we've already seen, but it also suggests a deeper reality. Mathematics represents the ultimate physical reality, showing us things we never would have expected without the mathematical models.
      4. Given that mathematicians get their ideas from the world around them, it's perhaps not surprising that mathematics fits physical phenomena. It may be surprising, however, that patterns pulled from one area apply to a totally unrelated area.
      5. As we'll see, the study of astronomical phenomena gave rise to trigonometric functions, but these trigonometric functions can also be applied to problems in mechanics, heat flow, and electricity and magnetism.
   B. The second paper, "The Unreasonable Effectiveness of Mathematics," was written by Richard Hamming in 1980 as a response to Wigner's article. Hamming was a prominent applied mathematician working in signal processing at Bell Labs.
      1. Hamming's article does a good job of clarifying the definition of mathematics. He describes the four faces of mathematics, two of which we've already talked about: number and geometry.
      2. The third face of mathematics is close reasoning. This tool helps mathematicians keep track of the multitude of patterns and abstractions they work with and reach legitimate conclusions. Formal logic, from the Greek tradition, is one way of expressing close reasoning.

3. The fourth face of mathematics is artistic taste. As we look for patterns, we find that there is an element of aesthetic appeal to good mathematics.

VI. These lectures will cover 4000 years in the history of mathematics. We will only be able to touch on the "big ideas."

A. The pivot point between the ancient Babylonians and Egyptians and modern mathematics occurs in the 17th century, when five different threads of mathematics come together—algebra, geometry, astronomy, mechanics, and the mathematics of motion.

B. We'll begin by looking at how different ancient civilizations—Mesopotamian, Egyptian, Greek, Indian, Chinese, and Islamic—developed these threads and laid the foundation on which western Europe would weave them together beginning around 1600.

C. We'll then spend five lectures on the 17th century, the point when these earlier mathematical ideas came to maturity and began to interact. One of the main figures in this interaction was Isaac Newton. His work *Mathematical Principles of Natural Philosophy* showed how all of the mathematics developed up to that point could be united.

D. Beyond the 17th century, we'll follow some of the most interesting developments and people in mathematics, with the intention of showing why it is so exciting to be a mathematician today.

Suggested Readings:

Courant, Robbins, and Stewart, *What Is Mathematics?*

Hamming, "The Unreasonable Effectiveness of Mathematics."

Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."

Questions to Consider:

1. Think of examples of quantities that are counted in discrete units (such as number of apples) and quantities that are measured in continuous units (such as quarts of applesauce). Speculate on how early people might have made the transition from numbers that count individual items to numbers that mark continuously varying and therefore necessarily arbitrary units.

2. Thinking back over your own mathematical education, identify as many points of contact as you can between algebra and geometry. When have you seen algebraic techniques used to illuminate geometric results; when have you seen geometric techniques used to illuminate algebraic results?

Lecture Two
Babylonian and Egyptian Mathematics

Scope: This lecture begins with an overview of the sources for the earliest recorded mathematics, from the Old Babylonian period (2000–1600 B.C.) in Mesopotamia and the Early Middle Kingdom in Egypt. Both civilizations had highly developed though different systems for recording and calculating with numbers, including fractions. Both civilizations knew how to find areas and volumes, including the area of a circle, and both used the Pythagorean theorem. The Babylonians demonstrated knowledge of how to generate arbitrarily many Pythagorean triples and had sophisticated techniques for calculating square roots and solving area problems that are equivalent to the solutions of quadratic equations.

Outline

I. In this lecture we will go back almost 4000 years, to the earliest recorded mathematics, which can be found in ancient Mesopotamia and Egypt.

A. Sophisticated mathematical texts were developed in both these civilizations at about the same time; these texts show us the origins of algebra and of the idea of approximating values.

B. In Mesopotamia, we will look at what is known as the Old Babylonian period, from roughly 2000 to 1600 B.C. In Egypt, we will look at the Early Middle Kingdom, dated to approximately the same period.

C. During this time, both of these civilizations became increasingly sophisticated. They had highly developed systems of commerce and were engaged in large-scale construction. They needed a large corps of people trained in basic mathematics to manage taxation and oversee building projects.

II. The records of mathematics we have from these ancient civilizations are the textbooks used to train such bureaucrats. These are primarily sets of problems with solutions.

A. From Egypt, we have only two problem books.

1. One of them, the Rhind Papyrus, was probably copied around the 17th century B.C. by a scribe named Ahmes. That scribe developed the text from a manuscript believed to have been written 200 years earlier. The papyrus roll is about 13 inches wide and 18 feet long.

2. The other important source for Egyptian mathematics from this period is the Moscow Mathematical Papyrus, probably written in the 19th or 18th century B.C. Unfortunately, only a small piece of this papyrus still exists today.

B. These two problem books show us how the ancient Egyptians represented number.

1. Our modern representation of number is quite sophisticated. We use 10 digits, the digits 0 through 9; if we want to represent a number that is greater than 9, we reuse these digits, but we put them in places that carry extra value.

a. For example, the two digits 1 and 3 can be combined to write 13 or 31, but the 3 may represent three 1s or three 10s, depending on where it is placed in the written representation.

b. In our modern system, the original 10 digits can be put in different positions to indicate the number of 10s, 100s, 1000s, and so on.

2. The system used in ancient Egypt was much simpler. This system represents numbers with strokes: 1 is a single stroke, 2 is 2 strokes, and so on.

a. We still see echoes of this in our modern representations of numbers. The written numeral 2, for example, is based on 2 horizontal strokes connected by a curve.

b. The stroke system becomes cumbersome in representing larger numbers. Thus, the Egyptians devised a system that replaced a group of 10 strokes with a single symbol called a *yoke*. A group of 10 yoke symbols was replaced with a different symbol, a *rope*.

c. A group of 10 rope symbols would be replaced with a lotus symbol to represent 1000, and other symbols were used for 10,000 and 100,000.

C. The Egyptians were also able to work with simple fractions by thinking of them as the reciprocals of integers.

1. The fraction $\frac{1}{2}$, for example, would be represented by 2 strokes with an oval drawn over the top of them.

2. A fraction like $\frac{3}{4}$ would be represented as $\frac{1}{2}$ plus $\frac{1}{4}$. This system was sufficient for the uses of fractions in ancient Egypt.
3. One of the problems from the Rhind Papyrus asks the following: We have a scoop and a certain volume of wheat called a *hekat*. If it takes $3\frac{1}{3}$ scoops to equal 1 *hekat*, what is the size of 1 scoop?

III. The ancient Egyptians were proficient in their work with geometry. They were able to find formulas for areas and volumes, and they knew the Pythagorean theorem.

A. The Pythagorean theorem is clearly much older than Pythagoras, who lived in the 6^{th} century B.C. Even by 2000 B.C., people in Mesopotamia and Egypt were conversant with the Pythagorean theorem.

1. The Pythagorean theorem relates to areas. To understand the theorem, imagine starting with a right-angle triangle and constructing a square on each of the three sides. The Pythagorean theorem states that the areas of the two smaller squares, when added together, are precisely the area of the large square.
2. Today, we state the Pythagorean theorem as: The square of the length of one of the legs of a right-angle triangle plus the square of the length of another leg of the triangle is equal to the square of the length of the hypotenuse ($a^2 + b^2 = c^2$).
3. The literal meaning of the word *square* in this statement helps us understand the relationship of the theorem to areas.

B. The Egyptians were also able to find the area of a circle.

1. The problem of finding areas of regions that are not square or rectangular was simplified to the problem of finding the side of a square whose area was equal to the given area. This approach, called *squaring a circle*, was used by Egyptian, Babylonian, Greek, medieval, and Renaissance mathematicians.
2. With this approach, we start with a circle of a certain diameter and then try to find the side length of a square that has the same area as that circle.

3. The Egyptians found that constructing a square that is $\frac{8}{9}$ the diameter of the original circle would give an extremely close approximation to the actual area of the circle.

IV. For the ancient Babylonian Kingdom in Mesopotamia, we have thousands of records of mathematics, but these are usually fairly small clay tablets, containing perhaps a single problem with its solution.

A. One of these tablets, known as Plimpton 322, is about 3.5 inches high and 5 inches long and is believed to have been created about 1800 B.C. This tablet lists cases of the Pythagorean theorem in which all three sides of the triangles are integers. Such cases are called *Pythagorean triples*.

1. The integers (3, 4, 5) make up one example of a Pythagorean triple ($3^2 + 4^2 = 5^2$). Other examples include (5, 12, 13) and (8, 15, 17).
2. Amazingly, among the triples listed on Plimpton 322 are (119, 120, 169), (3367, 3456, 4825), and (4601, 4800, 6649).
3. Unfortunately, Plimpton 322 consists only of this table of values. We have no idea of the procedure used to create it.

B. For small numbers, the Babylonians represented numbers in the same way that the Egyptians had.

1. A vertical wedge symbol was used to represent numbers up to 10, and a horizontal wedge represented 10. Rather than continuing up to 100, however, the Babylonians stopped at 60.
2. When they reached 60, instead of using a new symbol, the Babylonians reused the symbol for 1, in the same way that we reuse the symbol for 3 in the 10s place to mean 30.
3. Instead of our base-10 system, the Babylonians used a base-60 system. A symbol for 1 in the 60s place meant 60. A mathematician could then build up 60s to sixty 60s, then start over again with the symbol for 1 in the place of sixty 60s.
4. Thus, as we have a 1s place, a 10s place, and a 100s place, the Babylonians had a 1s place, a 60s place, and a 3600s place.
5. Look at the number 4601 as an example. There is one 3600 in 4601. Taking out 3600 leaves 1001. There are sixteen 60s (960) in 1001. Taking out 960 leaves 41. Thus, the number 4601 would be represented by one 3600, sixteen 60s, and forty-one 1s.

C. Another Babylonian tablet, known as YBC 7289, dates from between 1800 and 1600 B.C. and shows a representation of $\sqrt{2}$.

1. The problem this tablet explores is as follows: Consider a square and the diagonal of the square. If we take the length of the square as our unit, how many times can we measure off the length of that square against the diagonal?
2. If we try to perform the measurement, we realize that the length of the square will go into the diagonal once but not twice. How big is the piece of the diagonal that remains?
3. By breaking down the original unit, the length of the square, into groups of 60 progressively smaller subunits, the Babylonians were ultimately able to arrive at a very close approximation to $\sqrt{2}$. In our modern decimal representation, their solution is approximately 1.414213.

D. We can also get an idea of how the Babylonians viewed algebra problems from ancient clay tablets. In fact, the Babylonians did not think about algebra in the same way that we do today, but they did share common approaches with Greek, Indian, and Chinese mathematicians.

1. One of the problems from a Babylonian tablet is as follows: We have a square with a side of unknown length. We want to add to that the length of the square. The sum of these values is $\frac{3}{4}$; what is the length of the original square?
2. Today, we would find the solution with an algebraic equation: $x^2 + x = \frac{3}{4}$.
3. The Babylonians, however, thought about this problem geometrically, creating and manipulating rectangles to arrive at a method of proof that is exactly equivalent to the way we would solve this problem algebraically today. The technique for solving quadratic equations is called *completing the square*.
4. The Babylonian approach shows us the power of studying the history of mathematics. Learning to solve quadratic equations is easier if students are able to think about completing the square in terms of geometry and then encode that idea in algebraic notation.

V. After about 1600 B.C., the ancient sources for mathematics are not nearly as rich as they are for earlier periods. To some extent, mathematics seems to have gone underground at this point, taken over by the priestly castes for use in astronomical observations.

A. Nonetheless, we believe that the mathematical tradition became deeper and richer in both Mesopotamia and Egypt, largely because it would resurface in the Greek world, starting in about 600 B.C.

B. In the next lecture, we'll jump ahead to 600 B.C. to see how the Greeks used the ideas they inherited from Mesopotamia and Egypt.

Suggested Readings:

Buck, "Sherlock Holmes in Babylon."

Gillings, *Mathematics in the Time of the Pharaohs*.

Høyrup, *Lengths, Widths, Surfaces*.

Katz, *A History of Mathematics*, chap.1.

Van der Waerden, *Science Awakening I*, chaps. 1–3.

Questions to Consider:

1. The bureaucrats of the early 2nd millennium B.C. in Mesopotamia and Egypt needed to be skilled at engineering, law, and business. What are some of the mathematical skills they needed to perform their tasks?
2. We have no evidence of proofs in the modern mathematical sense in early Babylonian or Egyptian mathematics. What are the circumstances that would create a need for proofs?

Lecture Three

Greek Mathematics—Thales to Euclid

Scope: This lecture surveys more than 300 years of Greek mathematics, beginning with Thales of Miletus and continuing through Pythagoras to Zeno, Aristotle, Theaetetus, and Eudoxus. We will explore some of the mathematics developed by these thinkers, such as Thales's calculations of distances, the Pythagoreans' formulation of the idea of irrational numbers, Zeno's paradoxes, Aristotle's division of mathematics into *arithmos* and *logos*, Theaetetus's work with the Euclidean algorithm, and Eudoxus's procedure for determining that two irrational numbers are the same. We then turn to Euclid's *Elements*, considered to be the most important book of mathematics ever written. The *Elements* encapsulates earlier Greek mathematics, including solid geometry, properties of ratio and proportion, number theory, and prime numbers.

Outline

I. The Hellenistic world would dominate mathematical development from about 600 B.C. to A.D. 400. This lecture focuses on the period from 600 to 300 B.C., about the time when Euclid of Alexandria wrote his great books on mathematics.

A. Euclid is best known for the *Elements*, which is often considered to be a work of geometry but actually covers much more mathematics.

B. Scholars debate the extent to which the theorems recorded by Euclid were original to him.

C. Euclid's *Elements*, however, did summarize everything that was then known about mathematics and set the foundation for everything that would come afterward.

II. One of the earliest Greek mathematicians was Thales of Miletus (c. 624–c. 545 B.C.), who was interested in calculating distances between objects.

A. We know that Thales studied with Egyptian mathematicians and astronomers. He is said to have predicted an eclipse that occurred in the year 585 B.C. and calculated the height of the pyramids.

B. According to Plato, Thales also discovered a geometric method for determining the distance to a ship using two different points of observation on the shoreline. His fascination with the idea of distance set an important example for Greek mathematics.

1. Although we see some problems related to commerce and astronomy in Greek mathematics, the geometric interests of Thales would come to dominate.
2. This focus on geometry is evident in the dependence of the Greeks on the idea of ratio and proportion; often, instead of working with pure numbers, Greek mathematicians chose to work with ratios.

III. Ratios play a major role in the work of Pythagoras of Samos (c. 580–c. 500 B.C.).

A. Scholars believe that Pythagoras probably met Thales and may have even studied with him. He studied mathematics and astronomy in both Egypt and Babylon and was indoctrinated into many of the mystic practices of the Egyptian astrologer-priests.

B. After leaving Babylon, Pythagoras went to Italy, where he founded his own school based on many of the cult practices of Egyptian and Babylonian mathematicians.

1. The Pythagoreans believed that mathematics lay at the core of all nature. According to Plato, Pythagoras said, "At its deepest level, reality is mathematical in nature."
2. The Pythagoreans were particularly interested in expanding their knowledge of mathematics built around ratio and proportion. Certain developments in music and musical scales, as well as in architecture, are attributed to this group.
3. Of course, the Pythagorean theorem is often attributed to this group—but as mentioned in the last lecture, it was known much earlier. It appears, however, that the Pythagoreans were the first to develop a proof of the theorem.

C. The Pythagoreans are also credited with formulating the idea of irrational numbers—that is, numbers that cannot be represented using fractions or a ratio of two integers.

1. The first example of an irrational number we know of is $\sqrt{2}$, which is approximately equal to $\frac{7}{5}$. No fraction is exactly equal to $\sqrt{2}$.

2. The proof of the fact that $\sqrt{2}$ is not a ratio is often attributed to Pythagoras himself, although it may have been found by one of his disciples.

IV. Zeno of Elea (c. 495–c. 430 B.C.) is best known for his paradoxes.

A. Zeno was interested in questions related to the *continuum*, that is, what it means to consider all possible distances and to assign a number to any possible distance. The idea that every single point between two distances corresponds to some number presents a paradox.

B. Consider an arrow in flight and its velocity (distance traveled divided by time). If every point between the beginning and the endpoint of the arrow's trajectory is a number, can we assign a velocity to the arrow when it reaches a specific point?

1. Zeno believed that a velocity could not be assigned to the arrow because it doesn't travel any distance at any one instant in time.
2. To get from the starting point to the endpoint, the arrow must pass over the continuum—but at each point on the continuum, the arrow is not moving.
3. If at each point the arrow is not moving, how does it succeed in getting from where it is shot to where it lands?

V. Aristotle (384–322 B.C.) was one of the thinkers who attempted to circumvent Zeno's paradoxes by banishing them.

A. Aristotle also devised other separations that were significant for the development of mathematics. One of these is the difference between *arithmos* and *logos*.

B. *Arithmos* encompasses basic arithmetical calculations, such as those performed by people engaged in commerce or civil service.

C. *Logos*, which literally means "word," referred to closely reasoned, logical mathematical argument.

VI. Theaetetus of Athens (c. 417–369 B.C.) was associated with the Academy in Athens; his work is reflected in the writings of Euclid.

A. Theaetetus finished the description of the Platonic solids. These are solids whose faces are flat and are regular polygons. A cube is a simple example of a Platonic solid, each side of which is a square.

1. A tetrahedron is also a Platonic solid; it has 4 sides, each of which is an equilateral triangle. Other examples include the octahedron (8 triangles) and the dodecahedron (12 pentagons).
2. Theaetetus almost certainly discovered another Platonic solid, the icosahedron, which is made up of 20 equilateral triangles.

B. Theaetetus also worked on the recognition of rational numbers.

1. One of the problems encountered when working with rational numbers is that the same number can have very different appearances: $\frac{3}{4} = \frac{6}{8} = \frac{12}{16}$. We can see that these are all the same, but what if we had a more complicated fraction, such as $\frac{119}{91}$?
2. Theaetetus worked out a process known as the *Euclidean algorithm* for finding the greatest common divisor of two numbers. This process can be illustrated as follows:

- Start with $\frac{119}{91}$. Take out the largest integer, 1. The remainder is $\frac{28}{91}$.
- Take the reciprocal of that result, $\frac{91}{28}$.
- Take out the largest integer, 3. The remainder is $\frac{7}{28}$.
- Take the reciprocal of that result, $\frac{28}{7}$, which results in 4.

3. This process can also be reversed:

- Start with the last result, the integer 4. Take its reciprocal, $\frac{1}{4}$.
- Add the next integer, 3, to that result: $\frac{13}{4}$. Take its reciprocal, $\frac{4}{13}$.
- Add the next integer, 1, to that result: $\frac{17}{13}$, which is the original number, $\frac{119}{91}$, in reduced form.

4. Theaetetus realized that he could prove that $\sqrt{2}$ was not equal to a rational number by using the Euclidean algorithm combined with a geometric argument.

- Consider the length of a side of a square plus its diagonal: $1 + \sqrt{2}$.
- How many times does the side of the square go into that sum?
- That result is equal to 2 plus a number whose reciprocal is what we started with: $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$.
- This cannot be a rational number.

5. Working this process backward also gives an accurate method for approximating $\sqrt{2}$.

- Start with 2. Take the reciprocal and add 2: $2 + \frac{1}{2} = \frac{5}{2}$.
- Take the reciprocal of that result and add 2: $2 + \frac{2}{5} = \frac{12}{5}$.
- Continue this process to reach $\frac{169}{70}$. That result measures $1 + \sqrt{2}$ to an accuracy of 1 in 34,000.

VII. Eudoxus of Cnidus (b. c. 395–390 B.C.) was also a member of the Academy in Athens and interested in defining irrational numbers.

A. Theaetetus developed a procedure for determining that two rational numbers are equivalent to each other. Eudoxus worked out a procedure for determining that two irrational numbers are the same.

1. Eudoxus's procedure starts with two irrational numbers. If every rational number that is larger than one of these irrational numbers is also larger than the other one, and if every rational number that is smaller than one of these irrational numbers is also smaller than the other one, then the two irrational numbers must be the same.
2. Euclid picked up this idea and used it as a foundation for studying all numbers. He used the Euclidian algorithm for deciding whether two rational numbers are the same and Eudoxus's procedure for comparing the size of irrational numbers.

B. Eudoxus also calculated areas and volumes and is credited as the first mathematician to work out the *method of exhaustion*, which involves finding the area or volume of irregular regions or solids by dividing the region or solid into rectangular blocks.

VIII. Euclid of Alexandria, with whom we began this lecture, is a shadowy figure.

A. It is generally believed that Euclid lived roughly from 325 to 265 B.C., but in fact we do not have any definitive proof that he ever existed. No contemporary ever wrote about him, and the first biography was written 750 years after he lived.

B. However, the body of mathematical work that is attributed to Euclid, written around the year 300 B.C., was clearly written by one person; the writing is in the same voice with the same kinds of ideas and logical development.

C. Ptolemy I, the ruler of Egypt following the death of Alexander, built a center for scholars in Alexandria known as the Museion in the last years of the 4th century B.C. Euclid was almost certainly the leader of a group of mathematicians working at the Museion, tasked with collecting and explaining the mathematics known at the time.

1. We're not sure how much of the mathematics was original to Euclid, but his work is significant in the logical foundation and explanations he set down for the discipline.
2. He made important choices about what terms needed to be defined and how they would be defined; what ideas needed to be taken as postulates, assumptions, or axioms; and how these would then be structured into proofs.
3. Euclid's *Elements* is still used as a textbook today. It deals with a number of important topics, including ratio and proportion, similarities in geometry, number theory, geometric progressions, prime numbers, irrational numbers, various kinds of magnitudes, areas and volumes, solid geometry, and the Platonic solids.
4. Euclid wrote other texts in addition to the *Elements*. We have some portions of two other books on geometry, *Data* and *On Divisions*, and his book on optics. He is said to have written other books that no longer exist, including one on surface loci (two-dimensional curves) and another with the intriguing title *Porisms*.

IX. In the next lecture, we will look at the next 700 years in Greek mathematics, from the time of Euclid until about A.D. 400.

Suggested Readings:

Euclid, *Elements*.

Heath, *A History of Greek Mathematics*, vol. 1.

Katz, *A History of Mathematics*, chap. 2.

Van der Waerden, *Science Awakening I*, chaps. 4–6.

Questions to Consider:

1. Why is it much easier to conceive of equality between two integers or two ratios of integers than between two irrational quantities? Why would this problem be so important to Greek mathematicians?
2. What was it about the Greek approach to mathematics that led them to establish a central role for proof?

Lecture Four

Greek Mathematics—Archimedes to Hypatia

Scope: This lecture begins with Eratosthenes, who worked in the Museion established by Ptolemy I. Among his many accomplishments was the determination of a method for calculating the circumference of the Earth. From Eratosthenes, we move on to the greatest of all Hellenistic mathematicians, Archimedes of Syracuse. We'll look at two examples of his mathematics—a method for calculating the volume of a sphere and the computation of π to arbitrary precision. Another Hellenistic mathematician, Apollonius of Perga, wrote a complex and sophisticated text on conic sections that would prove to be indispensable to Isaac Newton almost 2000 years later. The last of the great Hellenistic mathematicians was Diophantus of Alexandria, who was the first person known to use a single letter to represent an unknown quantity, as we do in algebra today. We close the lecture with Hypatia of Alexandria, the first woman recorded to have made important contributions to mathematics.

Outline

I. This lecture picks up after Euclid and continues with Hellenistic mathematics to about A.D. 400. In the next lecture, we'll look at the astronomical work being done in the same period.

II. As mentioned in the last lecture, Ptolemy I established the Museion in Alexandria, which would become a center for mathematical work for the next 700 years. His successors, Ptolemy II and Ptolemy III, founded and stocked the Library of Alexandria, the greatest repository of books and scrolls in the ancient world.

A. One of the first mathematicians to follow Euclid in the Museion was Eratosthenes (276–194 B.C.), originally from Cyrene. Among his accomplishments was the discovery of a general method for finding all the primes below a given number. His method is known as the *sieve of Eratosthenes*, and computers searching for primes today rely on the idea behind his method.

B. Eratosthenes also measured the circumference of the Earth, which was known to be round. Observations of ships disappearing over the curve of the horizon and the shadow of the Earth projected onto the Moon during a lunar eclipse would have shown the shape of the planet. Eratosthenes devised a method for determining its circumference.

- Find a place where the Sun is directly overhead at a certain time of year.
- Then choose another place a certain distance north of the first location.
- Determine the angle made by the Sun's rays at the second point, and use this angle to determine the total circumference.

1. The Sun is so far away from the Earth that its rays hit the Earth essentially as parallel lines. The total circumference of the Earth can be determined using this knowledge.
2. Eratosthenes performed this calculation by digging two wells, one in southern Egypt and one near Alexandria. When the Sun was directly above the well in southern Egypt, he looked at the angle made by the rays at the well near Alexandria. He then accurately measured the distance between the two wells and determined the circumference of the Earth to be between 24,000 and 25,000 miles.

III. Archimedes of Syracuse (287–212 B.C.) was a colleague of Eratosthenes and the greatest of all the mathematicians in the Greek world.

A. Archimedes was known as an engineer and is credited with inventing the hydraulic screw. He was also instrumental in developing the idea of the block and tackle and discovered the power of levers. He famously said, "Give me a place to stand and I will move the Earth."

B. Archimedes turned his attention to machines of war, particularly catapults, as well as optics and the properties of mirrors. According to tradition, in 214 B.C., when the Roman General Marcellus was attacking Syracuse, Archimedes instructed the soldiers defending the city to line up with polished mirrors, focus them on Marcellus's ships, and set the ships ablaze.

C. The inventiveness of Archimedes was well-known to the Romans, and it was said that the Roman soldiers were terrified of his engines of war.

1. Marcellus eventually conquered Syracuse in 212 B.C. and, according to legend, instructed one of his soldiers to find Archimedes and bring him safely back to the Roman camp.
2. When the soldier found the great mathematician amid the chaos of the conquered city, Archimedes was deep in performing mathematical calculations on a sandboard.
3. The soldier ordered Archimedes to return with him to the camp, but when Archimedes said that he had to finish his calculations first, the soldier killed him.

D. Archimedes had asked that his tomb feature a sphere contained within a cylinder to commemorate his calculation of the volume of a sphere.

1. Archimedes's formula is that the volume of a sphere is $\frac{2}{3}$ the volume of a cylinder. This can be connected to the usual formula, $\frac{4}{3}\pi r^3$, as follows:

- Place a sphere inside a cylinder so that the circumference of the sphere exactly matches up with the circumference of the cylinder and the height of the sphere is equal to the height of the cylinder.
- The volume of the cylinder is the area of the base (πr^2) times the height of the cylinder, which is the diameter of the sphere $(2r)$. This produces the formula $2\pi r^3$.
- If the sphere is $\frac{2}{3}$ the volume of the cylinder, then the sphere is $\frac{2}{3} \times 2\pi r^3$, which is $\frac{4}{3}\pi r^3$.

2. Archimedes worked this formula out using the method of exhaustion developed by Eudoxus of Cnidus. In other words, he broke the volume of the sphere into thin slices and then added the volumes of all the slices to approximate the volume of the sphere.
3. This same idea is behind integral calculus, which at its heart deals with the problem of finding areas and volumes, and which works by this slicing mechanism.

E. Archimedes also found a way of approximating π, essentially to arbitrarily high accuracy.

1. The number π is the number of times that the diameter of a circle is used to measure off the circumference of the circle. The diameter goes around the circumference a little bit more than 3 times.
2. Archimedes began by considering a hexagon, a 6-sided figure made up of equilateral triangles that fits precisely into a circle. The circumference of the hexagon is 6 times the radius of the circle, or 3 times the diameter.
3. Archimedes reasoned that with a regular polygon inside of a circle, he could double the number of sides of the polygon and find the relationship between the new circumference and the old circumference.
4. Once he knew that a hexagon inside a circle has a circumference of 3 diameters, he was able to work out the circumference of a regular polygon with 12 sides, 24 sides, 48 sides, and 96 sides. Thus, he was able to measure π accurately to about three digits, or about 3.14.

IV. All of our evidence about work with conic sections in Hellenistic times comes from Apollonius of Perga (b. c. 260 B.C.).

A. Conic sections are curves obtained by slicing a pair of cones that are matched point to point.

1. If we make a horizontal slice through one of the cones, we get a circle. If we tip the plane that slices through the cone slightly, we get an ellipse. If we tilt the plane until it is actually parallel to one of the sides of the cone, the ellipse becomes a parabola.
2. If we continue to tilt the plane, it intersects the other cone, and we get one piece from the bottom cone and one piece from the upper cone. Those two pieces together give us a curve called a *hyperbola*.
3. All these basic curves, the circle, ellipse, parabola, and hyperbola, are united in this idea of slicing cones, and Apollonius explained the properties of these conic sections.

B. Apollonius's work on conic sections would eventually become an important part of analytic geometry and was used by Isaac Newton to explain the motion of the planets around the Sun. Today, we find conic sections easier to understand algebraically.

V. A number of important innovations come to us from Diophantus of Alexandria (c. A.D. 200–284).

A. Diophantus was the first person known to use a single letter to represent an unknown quantity, such as the variable x we use in algebra. He was known to Islamic mathematicians, the inventors of algebra, and he was responsible for a compact notation to represent quadratic polynomials—that is, polynomials that involve the square of an unknown plus a linear term plus a constant ($x^2 + bx + c$)—and cubic polynomials.

B. Diophantus is best known for his book *Arithmetica*, in which he studies problems with solutions that are only integers. Today, we call these *Diophantine equations*; classic examples of such equations are the Pythagorean triples.

1. If we have the equation $x^2 + y^2 = z^2$, and we let x and y stand for any positive numbers, we can always find a positive value for z.
2. The more interesting question is: Can we find integers for x and y that lead to an integer value for z? Solutions include those we saw earlier: (3, 4, 5), (5, 12, 13), and so on.
3. Diophantus was the first person to record a method for generating all the possible Pythagorean triples, although the Babylonians probably knew this method.

VI. The first woman to appear in the history of mathematics is Hypatia of Alexandria (c. A.D. 370–415).

A. Hypatia was the daughter of one of the scholars associated with the Museion, Theon of Alexandria. She was an accomplished mathematician and tutor. Although we have no original mathematics directly attributable to Hypatia, we know that she wrote commentaries on the work of Archimedes, Ptolemy, and Diophantus. In 1968, an Arabic translation of commentaries on Diophantus's *Arithmetica* was discovered, which is believed to have been written by Hypatia.

B. The Museion came to an end during the life of Hypatia. The last references to this center for scholarship occur in the late 4th century and probably coincide with the banning of all pagan temples by the Christian Emperor Theodosius I in A.D. 391.

C. Hypatia continued to live and work in Alexandria, but the arrival of the Christians ended the great period of mathematical achievement there.

1. Alexandria became embroiled in a power struggle for control of the city between Orestes, the prefect of Alexandria and the emperor's representative, and Cyril, the Christian bishop of the city.
2. Hypatia, a strong supporter of Orestes, was attacked and torn to pieces by a Christian mob on the streets of Alexandria in A.D. 415.

VII. In the next lecture, we'll turn to the development of astronomy in the Hellenistic world.

Suggested Readings:

Dijksterhuis, *Archimedes*.

Heath, *A History of Greek Mathematics*, vol. 2.

Katz, *A History of Mathematics*, chaps. 3, 5.

Stein, *Archimedes: What Did He Do Besides Cry Eureka?*

Van der Waerden, *Science Awakening I*, chaps. 7, 8.

Questions to Consider:

1. For Archimedes, would there have been practical uses for a highly accurate approximation to π? Why do you think he explored π to such a high degree of accuracy?
2. Diophantus was the first to use a single letter to represent an unknown quantity. Does this constitute the invention of algebra? If not, what more would be needed?

Lecture Five

Astronomy and the Origins of Trigonometry

Scope: Trigonometry was not applied to problems of surveying until the very end of the 16th century A.D. Its origins lie in problems of astronomy—in particular, the question of why the seasons are not the same length. Hipparchus of Rhodes, considered the father of trigonometry, answered this question by discovering a method for determining the length of the chord that connects the endpoints of a given arc of a circle. His methods were greatly refined by Ptolemy of Alexandria, for whom trigonometric relations and tables are at the core of his great astronomical work, the *Almagest*. In addition to its importance for astronomy and (eventually) surveying, trigonometry gave scientists their first real example of a function with a continuously varying input.

Outline

I. Astronomy was one of the dominant forces behind the development of mathematics, and in this lecture we'll see how astronomy led to the development of trigonometry.

A. Most people think of trigonometry in connection with land measurement—and today it is used in surveying—but it was originally applied to the study of astronomical phenomena.

B. We also think of trigonometry as being defined in terms of the ratio of the sides of a right-angle triangle, but again, that is not the way people have thought of the trigonometric functions—the sine, the cosine, and the tangent function—throughout most of the history of mathematics.

C. In this lecture, we will see that those functions emerge out of the problem of determining the length of a chord of a circle, that is, the straight-line distance between two points on a circle.

D. Trigonometry is extremely important because it introduces the function: the idea of a process that yields a well-defined output no matter what real number is input.

II. Let's begin by thinking about the solar system.

A. Most of us have a picture of the solar system in our minds: a bunch of balls, representing the planets, traveling in elliptical paths around a ball in the middle, representing the Sun.

B. Of course, no one has ever actually seen this picture of the solar system. What astronomers see is simply the night sky.

1. The first thing we observe by looking at the night sky is that the relative positions of the stars are fixed. This enables us to identify constellations: certain groupings of stars that always stay in the same relationship.

2. Even though the relative positions of the stars are fixed, the stars themselves move in the sky throughout the night. The one star that does not seem to move is the North Star, or the polar star, but all the others turn in a great celestial sphere around it.

3. Ancient observers noticed the fact that the Sun constantly changes position. They also realized that knowing the position of the Sun against the dome of the night sky is important for knowing when to plant, when to expect rains, when to harvest, and so on.

C. The path of the Sun is called the *ecliptic*, a term that comes from the means that astronomers found for determining the position of the Sun.

1. It is difficult to see where the Sun is against the stars because when the Sun is shining, the stars around it are not visible. Ancient astronomers used lunar eclipses to discern the path of the Sun.

2. When the Sun, the Earth, and the Moon are aligned, the Moon is at the exact opposite point in the sky from the Sun. During a lunar eclipse, the Moon is located where the Sun will be exactly six months in the future.

3. With this realization, astronomers were able to gradually plot out the points where the Sun would be at different times of the year. Collecting lunar eclipses over a long period of time enabled fairly accurate tracking of the Sun.

4. The signs of the zodiac are references to the constellations that give a rough idea of where the Sun is at different times of the year. An illustration in the frontispiece of Ptolemy's *Almagest* shows a model, called an *armillary sphere*, of the relative positions of the Sun, the various stars, and Earth, along with the signs of the zodiac.

D. Ancient observers also noticed that some stars move. In Greek, these became known as the *planetes*, or "wanderers," what today we call a *planet*. These stars also follow the ecliptic, the same path as the Sun. Astronomers tracked the planets because they believed that their positions might also have an important relationship to events on Earth.

III. Aristotle's view of an Earth-centered universe consisted of a great sphere of the stars, with the Sun and the planets traveling on the path of the ecliptic, but this model ran into some problems.

A. One of the first problems to be observed was that of *retrograde motion*, that is, the fact that the planets do not always move forward along the curve of the ecliptic. At times, they seem to slow down and back up.

B. We understand this phenomenon today because we realize that other planets, Mars, for example, circle the Sun, not the Earth. The Earth also circles the Sun, but because our planet is closer, it takes less time for Earth to travel around the Sun than it does Mars. At times, when Earth comes closest to Mars, we pass Mars, but from our perspective, Mars seems to be backing up. Once the Earth is far enough away, it looks like Mars has begun to move in its normal direction again.

C. The first person to find an explanation for the problem of retrograde motion was Aristarchus of Samos (c. 310–230 B.C.).

1. Aristarchus suggested that retrograde motion could be explained if both Mars and Earth circled the Sun, rather than Mars circling the Earth. His solution was considered preposterous.

2. People realized that if the Earth were circling the Sun, the planet would have to be traveling at incredible speeds. (In fact, the speed of the Earth around the Sun is about 67,000 miles an hour.) If our planet were traveling that fast, they thought we would have to be aware of it.

D. The solution to retrograde motion that was generally adopted came from Apollonius of Perga.

1. Apollonius suggested that rather than Mars circling the Earth, Mars is actually traveling on a smaller circle, called an *epicycle*, whose center circles the Earth.
2. This answer preserved one of the foundational Aristotelian viewpoints about the nature of the heavens—that all the events in the cosmos are based on circles.

IV. Another problem encountered in observation of the heavens is that the seasons are not all the same length. This fact was observed by Aristotle and explained by Hipparchus of Rhodes (190–120 B.C.).

A. The ancient astronomers knew the location of the Sun at both the summer and the winter solstice; these two points are exactly opposite each other on the great circle of the ecliptic. Drawing right angles from the line that connects the summer solstice to the winter solstice shows the vernal, or spring, equinox and the autumnal equinox. The seasons are marked by the time between these points.

B. If the Sun were traveling in a circle with the Earth at the center, each of the seasons should be the same length, but they are not. Winter, from the winter solstice to the spring equinox, is about 89 days long. Summer, from the summer solstice to the autumnal equinox, is more than 4 days longer.

C. Hipparchus of Rhodes saw that this difference could be explained if the position of the Earth was slightly off center in the universe. If the Earth is slightly off center and is closer to where the Sun is in winter, then the Sun actually travels a shorter arc in winter and a longer arc in summer.

D. Hipparchus's attempts to figure out the degree to which Earth was off center led to the problem of calculating the length of a chord, for which Hipparchus invented trigonometry.

1. Imagine that we have an arc of a circle and we want to figure out the length of the straight line that connects the two points at the ends of that arc.
2. First, we have to decide how we will measure the arc of a circle; we make this measurement in degrees, an idea that was inherited from the Babylonians.
   a. Today, we use degrees (°) to measure how much something has turned. If something has turned 90°, that means it has made a quarter-turn. This usage, however, is fairly recent.

   b. Up until the 18th century, degrees were a measure of arc length, which is a distance around the outer edge of a circle. The Babylonians divided the circle into 360°, almost certainly because that is approximately the number of days it takes the Earth to travel around the Sun.
   c. These degrees of arc length were then subdivided, according to the Babylonian system, into $\frac{1}{60}$ of a degree, which we call a *minute* (′). The minute is divided into $\frac{1}{60}$ of $\frac{1}{60}$ of a degree, or a *second* (″).
3. Given any particular arc length, we want to find the length of the chord that connects the endpoints of the arc.
4. That measurement will depend on the value of the radius. As we make the radius larger, the chord length changes; thus, the chord length is always described in terms of the radius.
5. If we have an arc of 90° (one-quarter of a circle) and we know the radius, the chord of 90° of arc is the radius multiplied by $\sqrt{2}$.
6. If we have a chord that corresponds to an arc of 60°, an equilateral triangle is formed by the two radial lines that go out to the circumference; thus, the chord of 60° is exactly 1 radius.
7. The Greeks were also able to figure out the exact value of a chord of 72°, which is $\sqrt{\frac{5-\sqrt{5}}{2}}$ times the radius.

E. The trigonometric functions come from these chords.

1. Think about an arc and the chord that connects the two endpoints of an arc. Using the terminology of bows and arrows, the arc is the bow and the chord is the string. The line from the center of the circle to the midpoint of the arc is the arrow. If we connect the radial lines that go from the center of the circle to the bottom of the arc and the top of the arc, the result looks like a bow and arrow.
2. If we consider not the full chord length but only half of the chord length (from the point where the chord attaches to the arc at the top down to the arrow), we are looking at the sine.

3. The Greeks did not work with the sine; as we'll see in the next lecture, this idea as well as the cosine came out of India. The tangent function was invented by Islamic astronomers.

V. The greatest astronomer and one of the greatest mathematicians of the Hellenistic period was Ptolemy of Alexandria (c. A.D. 100–c. 170), a scientist who shared the name of some of the rulers of Egypt.

A. Ptolemy put the ideas of Hipparchus into an incredible work on astronomy that runs to 13 books. His work explained astronomical phenomena, just as Euclid's *Elements* had explained mathematical principles. This great work of Ptolemy's was originally known as the *Mathematiki Syntaxis* (*Mathematical Collection*), but it has come down to us today as the *Almagest*.

B. In one part of the book, Ptolemy constructed a table of values that lists arc lengths and corresponding chord lengths.

1. Again, the circumference of a circle is 360°. It is easiest to work with chord lengths if the chord length is measured in the same units that are used to measure arc length.
2. With a circle of circumference 360°, the radius needed is $\frac{360}{2\pi}$, which is a little bit more than 57°. Ptolemy and other astronomers usually converted the circumference to $360 \times 60'$; then, the radius in minutes is approximately 3438.
3. Ptolemy explained how to use the chord lengths for two different arcs to find the chord length of their difference. For example, knowing the chord lengths for 72° and 60°, he could find the chord length for 12°.
4. Ptolemy also explained how to use the chord length for a given arc to find the chord length for half that arc. Thus, if we know the chord length for 12°, we can find the chord length for 6°, 3°, $\frac{3}{2}$°, and so on. This method does not, however, yield the chord length for 1°.
5. One of the advantages of using the same unit of measure for the arc length and the chord length is that as we work with smaller and smaller arcs, the ratio between the chord length and the arc length approaches 1. For very small arcs, we can assume that these two values are approximately equal.
6. This means that an arc of $\frac{3}{4}$° divided by a chord of $\frac{3}{4}$° should about equal an arc of 1° divided by a chord of 1°. This tells us that the chord of 1° is very close to the reciprocal of $\frac{3}{4}$ (which is $\frac{4}{3}$) times the chord of $\frac{3}{4}$.
7. Knowing the exact value of the chord of $\frac{3}{4}$, Ptolemy could then find a close approximation to the chord of 1°, and with that he could find the chord of $\frac{1}{2}$°. He then used his formula for the sum of two arc lengths to construct his table of chord lengths in intervals of $\frac{1}{2}$°. The chord length for $\frac{1}{2}$° corresponds to the sine of $\frac{1}{4}$°.

VI. These astronomical works that came out of the eastern Mediterranean would be imported to India, where astronomers would carry the ideas of Ptolemy much further and develop many new ideas in mathematics.

Suggested Readings:

Heath, *A History of Greek Mathematics*, chap. 17, 245–97.

Katz, *A History of Mathematics*, chap. 4, 135–57.

Questions to Consider:

1. Up through the time of Newton and into the early 18th century, degrees were considered a measurement of the length of an arc of a circle rather than a measurement of angle, though either is easily translated to the other. What are the advantages and disadvantages of thinking of degrees as a measurement of the arc of a circle?
2. A *radian* is a unit for measuring angles in which 2π represents a full turn of 360°. Why would radians be a natural unit to use if, instead of angles, we measure the length of the arc of a circle?

Lecture Six

Indian Mathematics—Trigonometry Blossoms

Scope: This lecture surveys early Indian mathematics, much of which is difficult to reconstruct because it was recorded as brief mnemonic descriptions written in verse in such texts as the Vedas and Sulbasutras. Indian methods for calculating and recording numbers were similar to those used in China. The great period of Indian mathematics occurred during the Gupta Empire, when astronomers with access to Alexandrian astronomical texts made significant advances in trigonometry and the related problem of finding formulas that would enable accurate interpolation of trigonometric tables. The great astronomical center of Ujjain would be the heart of Indian mathematics until the 13th century. This lecture concludes with a brief coda on the development of infinite series in Kerala during the 14th and 15th centuries.

Outline

I. The origins of Indian mathematics are rather uncertain.

A. The earliest records come to us from the Vedas, the epic poems that were written in the middle to late 2nd millennium B.C. The mathematics contained in the Vedas is sketchy and open to interpretation; often, we find only mnemonic devices for mathematics in these verses.

B. A collection of works that are appendices to the Vedas, known as the *Sulbasutras* (written c. 800–200 B.C.), describe how rituals were to be performed, lay out plans for constructing altars, and so on. These texts contain more mathematics than the Vedas.

1. The Sulbasutras explain how to find approximations to $\sqrt{2}$ and $\sqrt[3]{2}$ to enable priests to double the area and volume of an altar. Doubling the area requires multiplying the length and width by $\sqrt{2}$, and doubling the volume requires multiplying each of the three dimensions by $\sqrt[3]{2}$.
2. These poetic texts also contain references to the Pythagorean theorem, as well as interesting counting problems. One of the ways in which the texts were memorized was to recombine the syllables in the lines in different ways. Some of the earliest work in the field of mathematics called *combinatorics* comes from this endeavor.

C. The ancient Indian people used a system for recording numbers that is much closer to the system we use today.

1. The Indians had nine symbols for the digits 1 through 9, which could be combined with special symbols for the powers of 10. Under this system, to represent the number 327, for example, we would write the digit 3, the 100s symbol; the digit 2, the 10s symbol; and the digit 7.
2. This was not yet a full place-value system in the modern sense. The system had no digit for zero, which we use as a placeholder to mark the difference between, for example, 307 and 37.
3. As we will see in the next lecture, this system was also used by the Chinese, but scholars do not know where the system originated.

D. Zero was invented sometime between A.D. 300 and 600/700.

1. The earliest record we have of zero being used as a placeholder in a number is in a Hindu temple in Cambodia constructed in the year A.D. 683. As recorded in the Hindu system, the year of construction was a number that used the digit 0.
2. Zero had been used a bit earlier, not as a placeholder but as a number that could be manipulated, by the 7th-century astronomer Brahmagupta. He explained how to add or subtract zero and how to multiply by zero.

II. This lecture focuses on three great Indian astronomers: Aryabhata (476–550); Brahmagupta (598–c. 665); and Bhaskara Acharya, or Bhaskara the Scholar (1114–1185).

A. As mentioned in the last lecture, the astronomical works developed in Alexandria and elsewhere in the eastern Mediterranean would be taken up by Indian astronomers in the Kushan Empire.

1. The Kushan Empire had connections with an older Hellenistic empire, the Seleucids, who were based in Persia and central Asia.
2. Ptolemy's *Almagest* does not seem to have been among the astronomical texts brought to northern India during the Kushan Empire. Many other works of the same period,

however, were brought to India and translated into Sanskrit. Indian astronomers built on these texts, further developing the ideas of trigonometry.

B. The first of the great Indian astronomical texts is the *Surya Siddhanta* (written c. 4^{th} or 5^{th} century A.D.). Even in this early work, we see that the Greek chord has been replaced by the half-chord. The half-chord would become known in the West as the *sine*, a word that originates from an Arabic mistranslation of a transliteration of a Sanskrit word.

C. Indian astronomers also studied the cosine, called the *kotijya*.

1. Recall again the picture of the bow and arrow from the last lecture; the shaft of the arrow—the distance from the center of the circle to the chord—is the cosine.
2. The tip of the arrow between the chord and the arc of the bow, called by Indian astronomers the *ukramajya*, would become in English the *versed sine*. We no longer work with the versed sine, because it is simply the radius minus the cosine.

III. We turn now to the work of Aryabhata with trigonometric tables.

A. Aryabhata lived during the period of the Gupta Empire (A.D. 320–600) in Kusumapura, a center for astronomical work near the capital of Pataliputra, which today is modern Patna in India.

B. Aryabhata also worked with a circle of radius 3438, although he almost certainly was not aware of Ptolemy's *Almagest*. That text had found chord lengths for arcs down to $\frac{1}{2}°$, but Aryabhata worked with chord lengths in multiples of 3°45′.

1. In other words, he constructed his table by starting with a chord length for 60° and then found the chord lengths for 30°, 15°, $7\frac{1}{2}°$, and $3\frac{3}{4}°$ by taking half-angles.
2. Aryabhata's was a much rougher table than Ptolemy's, but he did important work with the problem of interpolation: How do we go about finding intermediate values within a table of values?
3. If we have a value that is halfway between two given values, a natural approach to interpolation is to take the output that is halfway between the two known outputs. Below is a table that will help us see how this idea works.

Input	Output
1	1
2	3
3	7
4	13

4. If our input is $2\frac{1}{2}$, what should the output be? The fact that $2\frac{1}{2}$ is halfway between 2 and 3 suggests that the output should be halfway between 3 and 7. The logical choice for the output is 5. This is called *linear interpolation*, which assumes that the inputs and outputs work as if they are on a straight line.
5. However, if we look at the output values, we see that the differences between them are increasing. The difference between 1 and 3 is 2; the difference between 3 and 7 is 4; and the difference between 7 and 13 is 6.
6. As we move further down the table, the differences increase; thus, the increase in the output going from 2 to $2\frac{1}{2}$ should not be as big as the increase in output from $2\frac{1}{2}$ to 3.
7. The following table agrees with the original table for integer inputs and gives us outputs at the half-integers so that the difference between the outputs increases by the same amount.

Input	Output
1	1
$1\frac{1}{2}$	$1\frac{3}{4}$
2	3
$2\frac{1}{2}$	$4\frac{3}{4}$
3	7
$3\frac{1}{2}$	$9\frac{3}{4}$
4	13

8. We can now see that a better output for an input of $2\frac{1}{2}$ is 3 plus an increase of $\frac{7}{4}$, or $4\frac{3}{4}$.

9. This approach is called *quadratic interpolation*. The Indian mathematician Brahmagupta showed how to do an arbitrary quadratic interpolation for any kind of input function.

IV. Brahmagupta was an extremely important mathematician and astronomer. He taught at the great astronomical center of Ujjain, which is located on the western edge of Madhya Pradesh, near modern Rajasthan.

A. Ujjain was founded in the 4^{th} century B.C. and would continue until the 12^{th} century. It was so important as a mathematical and astronomical center that it would mark zero longitude, or the first meridian, for Indian astronomers. The first meridian in our modern world is the line of Greenwich, chosen because of the royal astronomical observatory there.

B. As mentioned earlier, Brahmagupta was the first person known to have used zero and one of the first to use negative numbers. He realized the need to work with negative numbers and accept negative numbers as possible solutions.

C. The quadratic interpolation formula is often attributed to Newton, who extended it to interpolation by polynomials of higher degree. But Indian astronomers discovered this idea first.

D. Brahmagupta also did work on Diophantine equations, such as the Pythagorean triples we saw earlier ($3^2 + 4^2 = 5^2$).

1. Brahmagupta studied such Diophantine equations as $x^2 - 8y^2 = 1$. The ratio of any two integers that satisfy this equation gives a good approximation to $\sqrt{8}$.
2. Brahmagupta found a number of solutions to that equation: $3^2 - 8(1)^2 = 1$, $17^2 - 8(6)^2 = 1$, and $3363^2 - 8(1189)^2 = 1$.
3. Another example of a Diophantine equation studied by Brahmagupta is $x^2 - 61y^2 = 1$. One of his solutions was: $1{,}766{,}319{,}049^2 - 61(226{,}153{,}980)^2 = 1$.

V. Brahmagupta's ideas would further be developed by another great Indian astronomer, Bhaskara Acharya (1114–1185).

A. Bhaskara was interested in the problem of finding polynomials that interpolate values in tables of sines, cosines, and versed sines. The approach he came up with anticipated developments in western Europe that would come many hundreds of years later.

B. To find a polynomial to approximate the sine, Bhaskara considered the rate at which the sine function changes. In looking at the derivative of the sine function, he realized that the rate at which the sine function changes is given by the cosine function, and the rate at which the cosine function changes is given by the negative of the sine function.

C. Bhaskara saw how to use this fact to find what today we would call a *Taylor polynomial*, a quadratic polynomial that produces a good approximation to the sine function. If we know the exact values of the sine and cosine, we can work out this quadratic approximation.

D. Bhaskara's work marked an increase in the understanding of polynomials; up until his time, quadratic polynomials had been thought of as useful only for solving problems that involve areas. Cubic polynomials were thought of in terms of volumes.

1. Now that polynomials were used to approximate missing values in a table, there was no reason to stop at third-degree polynomials.
2. The Greeks stopped with third-degree polynomials because it made no sense to talk about a four-dimensional region, but with interpolation it makes sense to consider polynomials of degrees four, five, six, and so on.
3. It seems that this was one of the important sources of the development of the idea of a polynomial of general degree.

E. The type of Diophantine equation studied by Brahmagupta (e.g., $x^2 - 8y^2 = 1$) is known today as *Pell's equation*. Bhaskara found a procedure for solving Pell's equation no matter what multiplier is used for the second perfect square, as long as the multiplier itself is not a perfect square.

VI. Indian mathematics would continue after the Delhi caliphate conquered Ujjain in 1235, although the astronomical center was destroyed and the astronomers in Ujjain were dispersed.

A. The mathematical work of the time moved into southwestern India, in Kerala, where we find such astronomers as Madhava in the 14^{th} and early 15^{th} centuries, Paramesvara in the late 14^{th} century and into the 15^{th}, and Nilakantha in the 15^{th} century and into the early 16^{th}.

B. These astronomer-mathematicians furthered the general idea of Bhaskara of using the rate of change of a trigonometric function—and the rate of change of the rate of change—to find a quadratic polynomial that is a good approximation to the sine function, followed by a cubic polynomial, a fourth-degree polynomial, and eventually, a polynomial of arbitrary degree.

C. Madhava is generally credited with figuring out how to find polynomials of infinite degree, what today we call a *power series*. This is a representation of the function as a sum of powers of the unknown, with the powers going out to infinity.

D. Unfortunately, the astronomers who were working on power series in southern India did not communicate their results to anyone else. Because no other uses for the sine or cosine were known at this time, knowledge of the power series was lost. It was only in the 19^{th} century, when British archaeologists found some of these Indian texts, that mathematicians realized what these earlier thinkers had accomplished.

VII. In the next lecture, we'll move north from India to China to see what was happening there at the same time.

Suggested Readings:

Bressoud, "Was Calculus Invented in India?"

Datta and Singh, "Hindu Trigonometry."

Katz, *A History of Mathematics*, chap. 6, 210–32.

Varadarajan, *Algebra in Ancient and Modern Times*, 17–31.

Questions to Consider:

1. Today, trigonometric functions are usually introduced as ratios of sides of right triangles. Which do you consider to be conceptually simpler: the half-chord of a circle or the ratio of the opposite side to the hypotenuse in a right triangle?
2. By 1500, the most advanced mathematics in India—the infinite summation—was the preserve of a small group of devotees who explored it purely for its own sake. It died stillborn. Are there any lessons in this for us today?

Lecture Seven
Chinese Mathematics—Advances in Computation

Scope: The earliest surviving Chinese mathematical writing is from the Western Han dynasty. It contains indications of what was known in the 3^{rd} century B.C. and suggests a long mathematical tradition in China. Chinese mathematics was of a practical nature, and by the 1^{st} century A.D., mathematicians there were using advanced computational techniques. The Han mathematicians used a base-10 number system with decimal fractions and employed negative numbers. By the 5^{th} century A.D., Chinese mathematicians had found an approximation to π that would be the most accurate known for another 900 years. During the period A.D. 1000–1200, the Chinese did sophisticated algebra that included interpolation techniques and summation formulas that would not be rediscovered in Europe until the 17^{th} and 18^{th} centuries.

Outline

I. Thus far in the course, we've seen that mathematical ideas spring from different sources, including the fields of surveying and astronomy. Chinese mathematics grew primarily out of the needs of civil administration.

A. The record of mathematics in China begins with the early part of the Han dynasty (also known as the Western Han dynasty, or former Han dynasty, 206 B.C.–A.D. 25). This was the first Confucian dynasty, in which people were trained in mathematics, philosophy, and jurisprudence to become civil administrators.

B. Chinese textbooks consist primarily of problem sets, similar to the textbooks we saw in Babylon and Egypt.

1. Of the two important manuscripts we have from this time, the first is the *Zhou bi suan jing* (*Mathematical Classic of the Zhou Gnomon*, written c. 1^{st} century B.C.). This text contains problems of similar triangles related to surveying. The work also states the Pythagorean theorem and methods for calculating square roots.
2. The other manuscript is the *Jiuzhang suanshu*, or *Computational Prescriptions*. This text deals with finding areas and volumes and includes the Pythagorean theorem, as

well as linear equations and rules for calculating square roots and cube roots. The problems in this text involve surveying, commerce, and tax collection.

3. The same system for representing numbers that was used in India is used in the *Computational Prescriptions*, along with decimal fractions. The Chinese extended the idea of the units place, tens place, hundreds place, and thousands place in the other direction, looking at tenths, hundredths, and thousandths.
4. At about the same time, the 2nd or 3rd century B.C., negative numbers were also used as intermediate steps in some calculations.

C. Buddhism was introduced in China around the 1st century A.D., and shortly after that, we begin to find Indian astronomical texts in China. The Chinese began to work with trigonometric functions based on what they had learned from India. It would not be until about the year A.D. 1000, however, that the Chinese would begin to use zero as a placeholder.

II. The first Chinese mathematician we know by name is Liu Hui (fl. late 3rd century A.D.), who is best known for his work on surveying, recorded in the *Haidao suanjing*, or the *Sea Island Computational Canon*.

A. The title of the work comes from its first problem, which involves determining the height of a mountain on an island by an observer who is somewhere offshore and does not know his distance from the base of the mountain. Liu Hui explains how to use the idea of similar triangles to work out the height of the mountain.

B. Liu Hui's work also contains the first Chinese proof of the Pythagorean theorem. In addition, he explains a general method for finding areas and volumes that is equivalent to the Greek method of exhaustion.

C. Liu Hui used the same method for approximating π as Archimedes.

1. Both first found the circumference of a regular hexagon inscribed inside a circle and then doubled the number of sides.
2. Archimedes stopped when he got to a regular polygon of 96 sides, but Liu Hui continued to a regular polygon of 192 sides. He actually states that π is approximately 3.14, giving the tenths and hundredths in the decimal.
3. Liu Hui further stated that decimals could be expanded to as many places as needed, which may have been a new idea at the time. An approximation to the value of any distance could be found by using progressively smaller units.

D. The most accurate approximation to π from this time comes from Zu Chongzhi (fl. late 5th century A.D.). His approximation, $\frac{355}{113}$, is accurate to a degree of error of less than 3 parts in 10 million. He, too, may have found this approximation using polygons with progressively more sides.

III. Another problem the Chinese worked with during this period involves applying numbers to time.

A. When we try to apply numbers to time, we are working with different units—the day, the lunar month, and the solar year. How do we decide when these units line up?

B. Obviously, astronomers in Egypt and Babylon must have worked with this problem, but the first record we have of a solution for aligning the different cycles comes from a Chinese work called *Sunzi suanjing*, or *Sunzi's Computational Canon* (written c. A.D. 280–473). This solution today is called the *Chinese remainder theorem*.

1. The example given in the *Sunzi suanjing* is as follows: Consider three cycles of lengths 3, 5, and 7. Knowing that we are 2 units into the cycle of length 3, 3 units into the cycle of length 5, and 2 units into the cycle of length 7, how long ago were all these cycles lined up?
2. The situation is equivalent to finding a number that will have a remainder of 2 when divided by 3, a remainder of 3 when divided by 5, and a remainder of 2 when divided by 7.
3. Sunzi does not explain how he found the answer, 23.

IV. The next significant mathematician to come along was Li Chunfeng (A.D. 602–670), director of the astronomical observation service and chief astronomer and astrologer of the Tang dynasty (A.D. 618–907).

A. Li Chunfeng pulled together, corrected, and explained all the mathematical work from various texts that had been written in China up to the period of the Tang dynasty.

B. His work is known as *The Ten Computational Canons* (written A.D. 644–648).

V. Jia Xian, a court eunuch, was a Chinese mathematician from slightly after the year 1000 (fl. c. mid-11th century).

A. Jia Xian is generally credited with being the first person to come across Pascal's triangle, a triangular arrangement of numbers with 1s along each side and such that the numbers inside the triangle are obtained by adding the two numbers that are diagonally above them.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

B. Pascal's triangle is named for Blaise Pascal because of his work with it in the 17th century. It arises from the problem of expanding binomials.

- We begin with the binomial $1+x$.
- Multiply it by itself: $(1+x)(1+x)=1+2x+x^2$.
- Multiply that result by $1+x$: $1+3x+3x^2+x^3$.
- Multiply that result by $1+x$: $1+4x+6x^2+4x^3+x^4$.

The coefficients are exactly the numbers in Pascal's triangle, giving us an easy way to find the coefficients as we take progressively larger powers of $1+x$.

C. The reason Jia Xian studied these binomial expansions is that they offer a powerful way of approximating the value at which a polynomial is equal to zero, also known as the *root* of the polynomial.

D. For some polynomials, this value is an exact integer or, perhaps, a fraction. Most of the time, however, this value is an irrational number; thus, the best solution is a good decimal approximation to the number in question, which can be found using Pascal's triangle.

E. Jia Xian's ideas were further developed by Li Zhi (1192–1279), who wrote a work known as the *Ceyuan haijing*, meaning *Mirror Like the Ocean, Reflecting the Heaven of Calculations of Circles* (written 1248). This book contains many geometric techniques, but it also addresses the problem of finding roots of polynomials of arbitrarily high degree.

VI. The mathematician Qin Jiushao (c. 1202–1261) was the first person to explain the Chinese remainder theorem and to show how to use it in any situation with any number of cycles of any length.

A. One of the examples Qin Jiushao gives is as follows:

1. Find a number that is 32 units into a cycle of length 83, 70 units into a cycle of 110, and 30 units into a cycle of 135.
2. In other words, find the smallest positive number that gives a remainder of 32 when divided by 83, a remainder of 70 when divided by 110, and a remainder of 30 when divided by 135.

B. The number is 24,600.

VII. The culmination of Chinese mathematics came in the late 13th century, under the rule of Kublai Khan and slightly thereafter.

A. One of the great works from this time was the *Siyuan yujian*, the *Trustworthy Mirror of the Four Unknowns*, written by Zhu Shijie (c. 1260–1320) around 1303.

1. As mentioned in an earlier lecture, Isaac Newton is credited with discovering a general formula, known as the *Newton interpolation formula*, for interpolating polynomials using first differences, second differences, third differences, and so on.
2. However, the *Siyuan yujian* shows that Zhu Shijie and other Chinese mathematicians at the end of the 13th century clearly knew the Newton interpolation formula in its full generality.

B. Zhu Shijie also worked with Pascal's triangle to find the sum of the binomial coefficients along a diagonal. His result was later known as *Vandermonde's formula*, named after an 18th-century European mathematician.

C. Chinese mathematics began to disappear after the 13th century, probably because of the chaos that arose in China after the fall of Kublai Khan.

VIII. In the next lecture, we'll turn to Islamic mathematics and the emergence of algebra. We know that Islamic mathematics built heavily on work done in India, but how much of it drew on what was happening in China?

Suggested Readings:

Katz, *A History of Mathematics*, chap. 6, 192–210.

Martzloff, *A History of Chinese Mathematics*.

Needham, *Science and Civilisation in China*, 1–168.

Straffin, "Liu Hui and the First Golden Age of Chinese Mathematics."

Swetz, "The Evolution of Mathematics in Ancient China."

Questions to Consider:

1. Our earliest records for Chinese mathematics coincide with the creation of a sophisticated bureaucracy. Why would this favor the development of calculational techniques over the emphasis on logical reasoning that marked Greek and Hellenic mathematics?
2. Even though the Chinese were the first to use negative numbers in their calculations, they considered them to be computational conveniences rather than legitimate numbers. This would also be true of Islamic and European mathematicians well into the 17th century. What are the conceptual difficulties associated with negative numbers?

Lecture Eight

Islamic Mathematics—The Creation of Algebra

Scope: The development of mathematics in the Islamic world began in Baghdad under the ruler Harun al-Rashid, with the collection and translation of the scientific knowledge that then existed, drawing on the Hellenic, Indian, and Mesopotamian traditions. Algebra began here in the 9th century. The Islamic tradition encompasses many great mathematicians, including al-Kwarizmi, from whose writings we get the word *algebra* and whose name is the origin of our word *algorithm*; ibn Qurra, who made progress on Greek problems in number theory; al-Uqlidisi, whose work shows us the first use outside of China of decimals to represent fractions; al-Karaji, who set the stage for the study of arbitrary polynomials; al-Haytham, who advanced Archmidean techniques for computing areas and volumes; al-Samawal, who brought the study of algebra to a high art; and al-Khayyami, better known as Omar Khayyam, who studied cubic equations.

Outline

I. This lecture explores how the mathematical ideas that emerged in ancient Babylon, Egypt, the eastern Mediterranean, India, and China were drawn together in the Islamic world. Here, we will see the creation of algebra and advances in number theory, among other developments.

   A. The Islamic calendar begins in A.D. 622, when the Prophet Mohammed fled Mecca to Medina. We will focus on the Abbasid caliphate, which began around A.D. 750. A few years after the founding of this caliphate, Baghdad became the capital and cultural center of the Islamic world.

   B. The Islamic leader who would sponsor much of the mathematical activity in Baghdad was Harun al-Rashid (r. A.D. 786–809). He built a library of scientific works and had them translated into Arabic.

   C. Harun al-Rashid's successor, Abu Jafar al-Ma'mun (r. A.D. 813–833), established the Bayt-al-Hikma, or House of Wisdom, a center for scholarly work built around the library.

II. One of the first scholars in the House of Wisdom was Abu Jafar al-Kwarizmi (c. A.D. 790–840).

A. He wrote what is considered to be the first book of algebra in the history of mathematics, entitled *Condensed Book on the Calculation of Restoring and Comparing*.

1. As we've seen, Greek mathematicians worked on problems that we would think of as algebraic, and Diophantus introduced the idea of using a letter as a variable, but algebra itself did not yet exist.
2. Al-Kwarizmi introduced the idea of an algebraic equation with two quantities that involve an unknown, for example: $x^2 = 10x + 22$.
3. Al-Kwarizmi explained how to keep the equation in balance by adding, subtracting, or dividing by the same amount on both sides. This is a process he called "comparing and restoring"; the Arabic word for "restoring," *al-jabr*, is the origin of our word *algebra*.
4. Al-Kwarizmi's name is also the origin of our word *algorithm*, which is used in mathematics to mean a procedure with clearly prescribed steps.

B. Al-Kwarizmi's algebra didn't look like modern algebra. He did not use the kind of algebraic notation that we use today, and he expressed unknowns in words rather than variables. He also did not use the equal sign, which was not invented until the 16^{th} century in western Europe.

C. Al-Kwarizmi recognized the usefulness of the Hindu numeral system, which was a full decimal system that used zero as a placeholder. He helped to spread this representation of numbers across the Islamic world.

D. Most Islamic mathematicians at this time did not use negative numbers.

1. This can be traced in part to al-Kwarizmi, who still thought of the solutions to his algebraic equations geometrically, as the Babylonians and Greeks had. In thinking geometrically, it doesn't make sense to consider, for example, a negative length.
2. The avoidance of negative numbers made solving quadratic equations difficult. With an equation such as $x^2 + 10x = 39$, Islamic mathematicians resisted the idea of working with −39 in subtracting 39 from both sides and accepted only one of the two solutions to this equation, 3, while rejecting −13.

III. The mathematician Thabit ibn Qurra (A.D. 836–901) also lived in Baghdad and did important work in astronomy, geometry, mechanics, and number theory. Among his interests were *amicable numbers*.

A. If we add the proper divisors of 220—1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110—we get 284. If we then add the proper divisors of 284—1 + 2 + 4 + 71 + 142—we get 220. The Greeks called such pairs *amicable numbers*.

B. The Greeks also found another pair of amicable numbers, 1184 and 1210, but it was Thabit ibn Qurra who found the next pair, 17,296 and 18,416.

C. That knowledge eventually would be lost to Western mathematics and rediscovered by Leonhard Euler in the 18^{th} century.

IV. Abu'l Hasan al-Uqlidisi (fl. mid-10^{th} century) lived in Damascus and is best known for his book *Kitab al fusul fi al-hisab al Hindi*, or *The Book of Chapters on Hindu Arithmetic*. This book gives us the first use outside of China of decimals to represent fractions, shown by a hash mark over one of a series of numbers.

V. Abu Bekr al-Karaji (c. 980–c. 1030) was a mathematician living in Baghdad who wrote *al-Fakhri*, which means *The Marvelous*.

A. He explored the ideas of raising an unknown to an arbitrary power or a negative power and representing the reciprocal of the unknown by using a negative power (e.g., $x^{-2} = \left(\frac{1}{x}\right)^2$).

B. Al-Karaji saw that the way exponents combine is useful (e.g., $(x^3)(x^5) = x^8$) and stated the general rule for multiplying and dividing with exponents.

C. Al-Samawal, a 12^{th}-century Islamic mathematician, credited al-Karaji with being the first Islamic mathematician to discover Pascal's triangle, but we have no independent evidence of this discovery.

VI. One of the most interesting mathematicians from the Islamic world was Abu Ali al-Haytham, born in Basra, in southern Iraq, sometime around A.D. 965 (d. c. 1040).

A. Al-Haytham was an engineer in Basra, but sometime around the year 1000, he was enticed to Cairo, the capital of the Fatimid caliphate and the site of the al-Azhar mosque, founded in 975 as a center of learning.

1. One of the areas in which he concentrated his efforts was optics, trying to understand how mirrors of different shapes operate.
2. Al-Haytham published 92 scientific works, many dealing with optics, as well as other scientific and mathematical topics; 55 of al-Haytham's books still survive.

B. Al-Haytham worked with the Chinese remainder theorem and came up with an observation that would later be known as *Wilson's theorem*.

1. With a prime number (such as 7), the product of all the integers less than that prime (in this case, the integers 1 to 6) plus 1 will be divisible by the prime ($1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720 + 1 = 721 \div 7 = 103$).
2. For the prime number 11, the product of the integers 1 through 10 plus 1 is 3,628,801, which is exactly divisible by 11.

C. Al-Haytham also furthered Archimedes's ideas for finding areas and volumes and using the method of exhaustion.

1. Archimedes had solved the problem of finding the volume of a paraboloid. This bullet shape results from rotating a parabola around the line of symmetry.
2. Al-Haytham used half the parabola and rotated it around a vertical line to get a solid that comes to a point, a shape that is a common motif in Islamic mosque art.
3. For al-Haytham to find the volume of his solid of revolution, he had to find a formula for adding up the fourth powers of all the integers from 1 to n.
   a. The formula for finding the sum of the integers from 1 to n had been known since antiquity. The formula for the sum of the squares of the integers from 1 to n, which yields a cubic polynomial in n, was also known to Greek mathematicians.
   b. The sum of cubes involves a fourth-degree polynomial, a fact that may have been discovered by Greek mathematicians and was certainly known to Indian mathematicians by the middle of the 1st millennium A.D.

4. Al-Haytham found the fifth-degree polynomial that yields the sum of the fourth powers. He then came up with a general procedure to find the sum of fifth powers, sixth powers, and so on.

VII. Before we close, let's look briefly at two other important mathematicians who appeared in the Islamic world after al-Haytham.

A. The first of these is Ibn Yahya al-Samawal, a Jewish medical doctor who lived in the middle of the 12th century in Baghdad.

1. Al-Samawal understood the full power of polynomials and explained how to divide one polynomial by another, a process today called *synthetic division*.
2. Al-Samawal also worked with Pascal's triangle. He may have been the first person to discover the formula for finding the sum of second powers, third powers, fourth powers, and so on using the numbers in Pascal's triangle.

B. The poet Omar al-Khayyami (1048–1131) was also an accomplished mathematician. He was interested in the problem of finding the exact value of the roots of a cubic polynomial. Although he made progress on this problem, it would not be solved until 400 years later by Italian algebraists.

Suggested Readings:

Katz, *A History of Mathematics*, chap. 7.

———, "Ideas of Calculus in Islam and India."

Van der Waerden. *A History of Algebra*, chap. 1.

Questions to Consider:

1. Early Islamic algebra involved neither letters for the unknown quantities nor equations in the sense that we would represent them today. Instead, Islamic mathematicians described each "equation" using sentences. Al-Kwarizmi introduced the idea of keeping two expressions in balance as one seeks to isolate the unknown quantity. Why is this considered to be the beginning of algebra?
2. Why was this idea of keeping two expressions in balance such an important conceptual breakthrough?

Lecture Nine

Italian Algebraists Solve the Cubic

Scope: Islamic mathematics gradually spread into Europe, beginning with Leonardo of Pisa, also known as Fibonacci, in the 13th century. Fibonacci explained the Hindu-Arabic decimal system to an Italian audience in his book *Liber abaci* and worked with the sequence of integers that bears his name. Italian mathematicians began to make original contributions in the 16th century when they discovered methods for solving the general cubic and quartic equations. Such knowledge was closely guarded and used to defeat rivals in contests of mathematical prowess; thus, our story in this lecture is one of competition and intrigue. The colorful characters include del Ferro; Fontana, known as Tartaglia ("the Stammerer"); Cardano; and Bombelli. The solution of the general cubic equation would lead mathematicians to begin to work with imaginary numbers.

Outline

I. This lecture introduces us to the Italian algebraists, who learned mathematics from the Islamic scholars.

  A. One of the most significant insights of Islamic mathematics was the importance of the Hindu-Arabic decimal system, in which the power of 10 is determined by the place value of the digit.

  B. This Hindu-Arabic system of denoting numbers was used throughout the Islamic world, and by the 12th century, Italian merchants working in North Africa began to adopt it. They also learned Islamic methods in algebra, geometry, and other fields of mathematics.

II. One of the most important Italian mathematicians of this time was Leonardo of Pisa (c. 1170–1240), who referred to himself as a "son of Bonacci," or "Fibonacci," the name he is known by today.

  A. Fibonacci spent much of his early life in Algeria and devoted himself to learning mathematics, including algebra and geometry. Around 1200, he returned to Italy, where he published some of the most important early works in mathematics that would appear in western Europe.

  B. One of Fibonacci's first books was the *Liber abaci* (*Book of Calculations*), in which he explains the base-10 system for an Italian audience.

    1. The book also contains what we now call the *Fibonacci sequence*, the famous sequence of integers that begins 1, 1, 2, 3, 5, 8, 13. Each of the numbers in this sequence is the sum of the two previous numbers.

    2. Although it is called the Fibonacci sequence, this sequence was known before Fibonacci by Islamic, Indian, and Greek mathematicians.

    3. The Fibonacci sequence can be used to describe the number of seeds in the spirals on a sunflower head. If we follow the spirals around in two different directions, the two numbers will be consecutive Fibonacci numbers.

  C. In addition to writing his book on calculations, Fibonacci also wrote about geometry, algebra, and Diophantine equations. Recall that these are equations from number theory whose solutions are restricted to integers, such as the Pythagorean triples (in which the square of an integer plus the square of another integer is equal to the square of an integer).

III. Italy, especially northern Italy and the city of Bologna, would become an important center for the study of algebra. Bologna was the site of the first great western European university, which was founded around 1088.

  A. One of the mathematicians who taught at Bologna in the 15th and early 16th centuries was Scipione del Ferro (1465–1526), who was interested in the problem of finding the root of a cubic polynomial.

    1. One approach to this problem is to find an approximation to the value that satisfies a given polynomial equation (i.e., an approximation to the root). The ideal solution, however, would be to find an exact value for the root.

    2. If we're working with a quadratic polynomial (a polynomial of degree two), it's possible to find the exact value using square roots and the quadratic formula. Del Ferro sought an approach for finding the exact value for the root of a cubic polynomial (a polynomial of degree three).

    3. While a cubic polynomial might have more than one real root, we only need to find one root. If we find one root, then we can divide the polynomial by the *linear factor*—the factor of

degree one (x minus the root)—to reduce our polynomial to a quadratic for which the quadratic formula yields the roots.

4. Del Ferro found a method for determining the exact value of one of the roots of an arbitrary cubic polynomial, and he shared his discovery with Annibale della Nave and Antonio Fiore. Del Ferro's work marked the first time in the modern western European tradition that mathematics went beyond the accomplishments of the ancient Greeks.

B. Del Ferro's discovery was also important to mathematicians who sought patrons. Those who knew how to find the roots of a cubic polynomial had a great advantage in mathematical competitions sponsored by opposing patrons.

1. One such competition took place between Fiore and a mathematician named Niccolò Fontana, better known as Tartaglia, or "the Stammerer."
2. When Tartaglia heard that del Ferro had discovered a method for finding the roots of a cubic polynomial, he began to explore the same question and figured out the method. He then challenged Fiore to a competition and won.

IV. Another mathematician who became interested in the roots of cubic polynomials was Gerolamo Cardano (1501–1576), one of the greatest Italian algebraists of the 16th century.

A. Cardano was the son of a prominent law professor in Milan. He had a reputation as a gambler and did much of his early work on probability.

B. Cardano invited Tartaglia to come to Milan and share the secret of how to find the root of a cubic polynomial. Tartaglia agreed but asked Cardano to keep the method a secret because he wanted to use it to find patronage for himself. Cardano later learned from Annibale della Nave that the "secret" was known by others.

C. In 1545, Cardano published his great work in algebra, the *Ars magna* (*Great Art*), which included the method for finding the root of a cubic polynomial.

D. At about the same time, Lodovico Ferrari (1522–1565) was sent to Cardano as a servant. Cardano recognized Ferrari's mathematical talent and made him his secretary. Ferrari quickly went beyond what Cardano had accomplished in mathematics, extending the

formula for the root of a cubic polynomial to find the exact value of the root of an arbitrary polynomial of degree four.

E. Shortly thereafter, Ferrari bested Tartaglia in a competition to win an academic position in Brescia. By the end of the first day, Tartaglia was clearly falling behind and fled the competition in disgrace. Ferrari later became the tax assessor for the city of Milan.

V. Let's look for a moment at the method for finding the root of an arbitrary cubic polynomial.

A. We begin with a polynomial equation of degree three [$x^3 + \alpha x^2 + \beta x + \gamma = 0$]. If we replace x by $x - \alpha/3$, we can write this cubic equation in the form $x^3 + cx = d$. As an example, let's use the cubic equation $x^3 + 6x = 4$.

B. We use the constant, 4, and the coefficient of x, 6. We divide 6 by 3, then cube the result: $2^3 = 2 \times 2 \times 2 = 8$. Our two key numbers for this equation, then, are 4 and 8.

C. The key to finding a root of this cubic equation is to find two numbers whose difference is 4 and whose product is 8. This problem can be restated in terms of solving a quadratic polynomial, and we know how to find the exact value of the solution of a quadratic polynomial.

D. In this case, the two numbers whose difference is 4 and whose product is 8 are $2 + 2\sqrt{3}$ and $-2 + 2\sqrt{3}$. The difference of those two values is 4, and the product is 8.

E. The difference of the cube roots of $2 + 2\sqrt{3}$ and $-2 + 2\sqrt{3}$ [$\sqrt[3]{2 + 2\sqrt{3}} - \sqrt[3]{-2 + 2\sqrt{3}}$] gives the root of the original equation.

VI. Another famous Italian mathematician was Rafael Bombelli (1526–1572), an engineer born in Bologna.

A. Early in his career, Bombelli was commissioned to drain the marshes in the Val di Chiana, a high mountain valley north of Rome. The project stretched over nine years, during the course of which Bombelli read Cardano's book on algebra and began to write his own treatise on the same subject.

B. When the project at the Val di Chiana was finally complete, Bombelli was brought to Rome to take on several engineering projects for the pope, including the restoration of the Santa Maria Bridge, an ancient bridge spanning the Tiber, and the draining of the Pontine Marshes. Neither of these projects was completed successfully.

C. While he was in Rome, Bombelli discovered the *Arithmetica* of Diophantus. He began to translate this work and to incorporate it into his own book on algebra. Eventually, Bombelli produced an important work in algebra that would influence algebraists throughout western Europe.

D. Bombelli realized the importance of working with the square roots of negative numbers. Let's find the root of another cubic equation as an example: $x^3 - 15x = 4$.

1. The two key numbers that we will work with in this case are 4 and $-\left(\frac{15}{3}\right)^3$ (which equals −125). We want to find two numbers whose difference is 4 and whose product is −125.
2. We can't find two real numbers whose difference is 4 and whose product is −125, but in fact there is a root to this cubic polynomial: $4^3 - 15(4) = 4$.
3. The method of del Ferro and Cardano doesn't work in this case, but the solution can be found by working with the square roots of negative numbers. Bombelli was the first to try this approach.
4. Bombelli showed that if we allow square roots of negative numbers, then $2+\sqrt{-121}$ and $-2+\sqrt{-121}$ are two numbers whose difference is 4 and whose product is −125.
5. The cube roots of $2+\sqrt{-121}$ and $-2+\sqrt{-121}$ are $2+\sqrt{-1}$ and $-2+\sqrt{-1}$, respectively, and their difference is 4.
6. Descartes would later call these square roots of negative numbers *imaginary numbers*. In the 18th century, imaginary numbers became critical to advances in mathematics. The numbers that are built out of real and imaginary numbers, such as $2+\sqrt{-1}$, are called *complex numbers*.

VII. By the end of the 16th century, algebra was fairly well understood. In the next five lectures, we'll move into the 17th century, which is the pivot for this entire series. In Lecture Ten, we'll return to the subject of astronomy, which motivated much of the mathematics developed during the 17th century, particularly the invention of the logarithm by John Napier.

Suggested Readings:

Gindikin, *Tales of Mathematicians and Physicists*, 1–26.

Katz, *A History of Mathematics*, chaps. 8, 9.

Nordgaard, "Sidelights on the Cardano-Tartaglia Controversy."

Van der Waerden, *A History of Algebra*, chap. 2.

Varadarajan, *Algebra in Ancient and Modern Times*, 47–92.

Questions to Consider:

1. Hellenistic mathematicians worked with cubic equations, but always expressed in terms of volumes. Was the Islamic innovation of formal equations needed before the general solution of these cubic equations could be found?
2. Who, if anyone, was in the right in the Cardano-Tartaglia dispute?

Lecture Ten
Napier and the Natural Logarithm

Scope: In this lecture, we look in detail at logarithms, a tool invented to assist astronomers in making accurate calculations from their observational data of the heavens. Two astronomers in particular, Tycho Brahe and Johannes Kepler, sought to improve Copernicus's earlier model of the universe, in which he posited that the planets move around the Sun in circles. Kepler eventually discovered that the paths of the planets are elliptical rather than circular. John Napier was a Scottish nobleman who became interested in finding a computational tool that would exploit the power of exponents to facilitate astronomical calculations. He created tables of logarithms that were accurate to 7 digits, a remarkable result equivalent to calculating the first 23 million powers of 1.0000001. Napier's logarithms would later play an important role in calculus.

Outline

I. This lecture explores the most important functions to be discovered since the trigonometric functions—the logarithm and the exponential function. Before we look at these topics, however, we need to see what was happening in astronomy in the middle of the 16th century because developments at that time would influence 17th-century mathematics.

II. The year 1543 saw the publication by Nicolaus Copernicus (1473–1543) of his groundbreaking work, *On the Revolutions of the Heavenly Spheres*.

A. Copernicus was a Catholic monk in Poland and was interested in the motion of the planets. In particular, he was disturbed by the problem of retrograde motion that we discussed in Lecture Five—that is, the apparent backward motion of the planets at certain times.

1. Archimedes had raised this problem, and Aristarchus had suggested the idea of the Earth traveling around the Sun as an explanation of retrograde motion. That explanation was dismissed because we on Earth have no sense of the planet's motion.

2. Slightly later, Apollonius of Perga offered the theory of epicycles—that planets move in circles whose centers then travel around the Earth—to explain retrograde motion.
3. This explanation became the foundation for what was known as *Ptolemaic astronomy*, named after the Greek astronomer Ptolemy, who showed how the movement of the planets could be modeled using epicycles and equants.
4. Over the succeeding centuries, the system became more and more complicated because it never quite matched what people observed in the heavens. Epicycles were added to epicycles.

B. Copernicus sought to simplify the model by going back to the idea of Aristarchus's. He postulated that, in fact, the Earth travels around the Sun. The problem with Copernicus's work was that he also assumed that the planets travel around the Sun in a circle, but observations were sufficiently accurate by this time to show that this model does not accurately predict the locations of the planets.

III. To improve the model, more accurate observations of the positions of the planets were needed; the man who undertook to make these observations was a wealthy Danish astronomer named Tycho Brahe (1546–1601).

A. Brahe, who had studied astronomy in Copenhagen and Germany, set up an astronomical observatory on the island of Hven, between what are now Sweden and Denmark.

1. Brahe made two important observations in Hven. The first, in 1572, was the observation of a supernova, which revealed that the stars are not eternal. This discovery undermined the Aristotelian view of the heavens.
2. Brahe also observed and measured the path of a comet in 1577. Comets were once believed to be phenomena that were confined to the atmosphere of the Earth, but Brahe showed that the comet he observed was well beyond the Moon. Again, the fact of something moving through the heavens revealed that the universe is not perfect and unchanging.

B. Brahe made all his observations without a telescope. The best one can usually hope for in this case is an accuracy of about 2′ of arc (360° is a full circle, and $\frac{1}{60}{}^{\circ}$ is a minute). Differentiating between two stars in the sky separated by less than 2′ of arc is difficult, but

Brahe was sometimes able to fix the positions of the planets to within 1′ or even $\frac{1}{2}$′ of arc.

C. In 1599, Brahe was hired as the court astrologer to the Holy Roman Emperor Rudolf II, whose capital was in Prague. Shortly after he took the position, he hired a young assistant, Johannes Kepler (1571–1630).

1. In 1601, Brahe died from what was later determined to be mercury poisoning.
2. It has been suggested that Kepler had a hand in Brahe's death. He benefited from it, inheriting both Brahe's position as court astrologer and his valuable data.

D. It became Kepler's life work to find a model that would fit these meticulous data. Eventually, he realized that the reason the model of Copernicus's was off is that the orbits of the planets are not circles but ellipses, and the Sun is located at one focus of the ellipse.

E. Kepler also established other laws of celestial motion. He realized that the planets sweep out equal area in equal time. This means that the planets speed up when they come close to the Sun and slow down when they move away.

1. Another one of Kepler's laws is that the square of the time required to complete an entire revolution of the Sun is proportional to the cube of the average distance to the Sun.
2. In many respects, Kepler was an astrologer and a mystic. He explained the relative distances of the planets in terms of Platonic solids, each one sitting inside another. He believed that the crystalline spheres in which the planets were embedded rubbed against each other, and he described the music of the spheres.

IV. With the data he inherited from Brahe, Kepler needed to perform calculations that were accurate to at least 5 digits.

A. Shortly after Kepler began working with Brahe's data, the telescope was invented, and Kepler was able to accumulate even more accurate data. Soon, astronomical calculations required 7 digits of accuracy, and by 1630 they required 10 digits of accuracy.

B. The Scottish nobleman John Napier (1550–1617) invented a computation device to assist in performing multiplication and

division and evaluating trigonometric functions to 7 or 10 digits of accuracy.

1. Napier had wide-ranging interests. One of his most famous writings was a diatribe against the Catholic Church, in which he theorizes that the revelation of St. John depicts the pope as the Antichrist.
2. Napier traveled to Italy in 1594 and may have met Galileo. He returned from Italy with an interest in developing computational tools to facilitate difficult multiplication and division problems.
3. In 1614, Napier published his work, *Description of the Wonderful Canon of Logarithms*.

C. Napier may have invented the word *logarithm* by combining two Greek words, *logos* and *arithmos*. As mentioned in an earlier lecture, the Greeks had separated mathematics into two different spheres; one was *logos*, or logical mathematical reasoning, and the other was *arithmos*, or calculation. The beauty of the logarithm is that it applies logical mathematical reasoning to calculation.

D. The idea behind the logarithm is to exploit the power of exponents. Let's look at a simple example.

1. If we want to multiply 16×128, we know that 16 is 2^4 and 128 is 2^7. Earlier Islamic mathematicians had known that multiplying $2^4 \times 2^7$ can be accomplished by adding the exponents: $2^{11} = 2048$.
2. What if we want to multiply 19×33? There is a power of 2 that gives us 19; it falls between 2^4, which is 16, and 2^5, which is 32. The exact value is 4.247928.
3. In the same way, we can write 33 as a power of 2, and we can multiply 19×33 by adding the two exponents. The result is $2^{9.292322}$, which is approximately 627. Because 19×33 is an integer, we know that it will be exactly 627.
4. We started with 19, and we wanted to find the exponent of 2 that would give us 19. Napier called the exponent the *logarithm* of 19. The number that we exponentiated is called the *base*. Thus, 2 is the base, and 4.247928 is the logarithm of 19 for the base 2.

5. The problem then becomes: Given an integer, find the exponent of 2 that will lead to that integer; in other words, find the logarithm of that integer for the base 2.
6. This introduces the idea of the exponential function. We're not just interested in the integer powers of 2 or the simple fractional powers of 2. We're now interested in 2 raised to a power that could be any real number.

E. One way to calculate logarithms to an accuracy of 7 digits is to calculate the first 23 million powers of 1.0000001.
   1. Napier realized that performing those calculations was an impossible task, but his insight that a simple conversion factor could be used to find logarithms for any base once they were known for another base enabled him to find the 23 million values easily.
   2. He found the first 230 powers of 1.01, which is roughly equivalent to finding 230 powers of 1.0000001 that are multiples of 100,000. He then interpolated between those, finding powers that are multiples of 2000, then multiplies of 100.
   3. With just a few hundred calculations, Napier was able to set up tables that would enable mathematicians to find logarithms to the base 1.0000001 for all numbers between 1 and 10, yielding the desired 7-digit accuracy.

F. Because Napier could change bases so easily, he realized that there was no need to stick with the base 1.0000001. The question now became what base to use.
   1. Consider two lines with points moving along each line. On the first line is the point that we want to take the logarithm of. On the second line, we look at the image of that point under the logarithm.
   2. For example, for the point 3 on the first line, the position on the second line is the logarithm of 3. When the point moves to 5 on the first line, the point on the second line moves to the logarithm of 5.
   3. If the point on the first line travels at a uniform speed, how fast does the point on the second line move? Napier saw that the point on the second line moves at a speed that is inversely proportional to the distance that the point on the first line has moved.

   4. In other words, if the first point has moved to position x, the speed of the point on the second line is some constant divided by x. The value of the constant depends on the base that we are using for the logarithm.
   5. Napier chose the logarithm that would give him a speed of exactly $1/x$. Today, we call this the *natural logarithm*.
   6. Napier never worked out the value of this base, but it was later observed to be about 2.71. This number is referred to as e, which is called the *base of the natural logarithm*.
   7. Another mathematician, Henry Briggs, in London, realized that a base of 10 would be much easier to work with. He would publish his own table of logarithms that was accurate to 10 digits.
   8. Napier's logarithm, however, would have an important role to play in calculus. Two Belgian Jesuits discovered that the area underneath the curve of a hyperbola from 1 to any value t is given by Napier's natural logarithm of t.

V. In the next lecture, we'll return to the early 1600s to look at the work of Galileo and the beginnings of the mathematics of motion.

Suggested Readings:

Katz, *A History of Mathematics*, 416–20.

Toeplitz, *The Calculus*, 86–94.

Whiteside, "Patterns of Mathematical Thought," 214–31.

Questions to Consider:

1. In what ways do Napier's logarithms combine the two Greek conceptions of mathematics, *logos*, or logical reasoning, and *arithmos*, or calculation?
2. The natural logarithm fascinated scientists of the 17th century because it gave them another example of what today we would call a function, a well-defined rule for which each input value produces a unique output value. What is special about this function that distinguishes it from the other functions, especially the trigonometric functions, that were known at the time?

Lecture Eleven

Galileo and the Mathematics of Motion

Scope: This lecture explores the work of Galileo Galilei, an acknowledged great thinker, but one whose precise contributions are difficult to pin down. Much of his work was foreshadowed by earlier scientists, and much of it had to be revised or completed by later scientists. Galileo's greatness can be located in his ability to ask the right questions and his understanding that the answers could be found in mathematics. Galileo believed that the Earth travels around the Sun and sought to explain why, if that was the case, people on Earth don't sense the planet's motion. To this end, he studied gravity and the laws of motion. He explored velocities, partially worked out the concept of inertia, and anticipated the method of finding total distance traveled by calculating the area under a curve that represents velocity. Galileo's reliance on mathematical models to explore physical phenomena would lead to further discoveries in science and the development of analytic geometry and calculus.

Outline

I. In Lecture One, we mentioned that five strands of mathematics would come together in the 17th century: algebra, geometry, astronomy or astrology, mechanics, and the mathematics of motion. The first figure in which we see this fusion is the great scientist Galileo Galilei (1564–1642).

A. Galileo is universally acknowledged to be an important scientist, but as E. J. Dijksterhuis, one of the great historians of science in the 20th century, said of him, "On the question of what precisely his contribution was, and wherein his greatness essentially lay, there seems to be no unanimity at all."

B. Virtually everything Galileo did was foreshadowed—in some cases, fully realized—in the work of earlier scientists, and much of his work had to be revised or completed by later scientists. His greatness lay in his talent for asking the right questions and knowing where to look for solutions.

C. In particular, Galileo knew that the key to understanding the motions of the planets (celestial mechanics) could be found in mathematical modeling. This emphasis on mathematics set Galileo apart from other thinkers of his day, and his work would eventually lead to the development of calculus.

II. Galileo was trained as an algebraist. He got his first job in 1585 in Florence and would hold a succession of positions in Sienna, Vallombrosa, and Pisa. In 1592, he was appointed to a position in Padua, where he may have met John Napier in 1594. In 1610, he published *The Starry Messenger*.

A. Some years earlier, Galileo had gotten hold of one of the first telescopes, originally built in the Netherlands. He used it to see not only objects that were far away on Earth but also the Moon and the other planets in the heavens. He was the first to observe moons orbiting other planets and geographical features on the face of the Moon.

B. *The Starry Messenger* was a popular book that laid out Galileo's arguments for the fact that the Earth orbits the Sun rather than the other way around. He also made this point quite clearly in a later book, *Letter to the Grand Duchess* (1616).

C. After the publication of *Letter*, Galileo was brought before the Inquisition in Rome and warned not to state categorically that the Earth traveled around the Sun. He could use that idea as a basis for a model, but he was not to assert it as fact because it contradicted passages in the Old Testament in which the Earth was described as stationary.

D. Galileo was convinced that the Earth in fact moved. To support that view, he knew he had to explain why people on Earth were not aware of the planet's motion. Thus, he began to try to model gravity, inertia, and other aspects of physics that would explain how we could live on a moving Earth without being aware of its motion.

E. Galileo's book explaining his understanding of the universe appeared in 1632: *The Dialogue Concerning the Two Chief Systems of the World.*

1. The book is written as a debate among three protagonists concerning the issue of whether the Earth travels around the Sun or vice versa.

2. The three protagonists are Salviati, Sagredo, and Simplicio. Salviati is an intelligent layman who is trying to learn the truth, Sagredo argues that the Earth travels around the Sun, and Simplicio holds to the belief that the Earth is stationary.
3. Although the book claimed to be impartial, it clearly argued in favor of the idea that the Earth orbited the Sun. For this reason, Galileo was again called before the Inquisition in Rome.
4. Galileo recanted and was forbidden from publishing *Two Chief Systems of the World*. He was restricted to house arrest at his home in Arcetri for the rest of his life.

III. Galileo realized that the key to understanding why we don't sense the motion of the Earth is gravity.

A. To Aristotle and later scientists/philosophers, gravity (*gravitas*) was a property inherent in certain bodies—the tendency of certain objects to move toward the center of the universe.

1. The Earth was at the center of the universe because it had more gravity than anything else.
2. As opposed to gravity, an object could have *levity*—a tendency to move away from the center of the universe. Fire rises because it has levity.

B. We can see one of the problems with this theory of gravity and levity in the act of throwing a ball. When a ball is thrown, it first moves away from Earth. Why doesn't it immediately start moving toward the center of the universe?

1. One explanation that was offered was the *theory of impetus*—the idea that a moving object accumulates an impetus, or tendency to keep moving in a given direction.
2. Medieval philosophers were able to explain the motion of a thrown object by postulating that the object acquires a certain impetus from the hand; they expected the impetus to gradually dissipate as the object traveled through the air and finally fell to the ground.
3. Early explanations of the motion of cannonballs used this idea. We have illustrations showing a cannonball being fired from a cannon, traveling in a straight line until it runs out of impetus, and finally falling vertically to the ground. Clearly, however, a cannonball travels along an arc.

C. One of the properties of gravity tested by Galileo was the idea that a heavy object falls faster than a lighter object.

1. We often hear that Galileo went to the top of a tower, dropped two balls of unequal weight, and observed that they hit the ground at the same time. We have no evidence that Galileo ever conducted this experiment, but it was conducted by the Belgian-Dutch scientist Simon Stevin (1548–1620) in 1586.
2. Stevin is an important figure for a number of reasons. In 1585, he wrote *La Thiende*, a book advocating the adoption of the Islamic system of decimal fractions—tenths, hundredths, thousandths, and so on—over the Babylonian system that used sixtieths. At the time, the two systems existed side by side, with merchants using decimal fractions and scientists using sixtieths.
3. Thomas Jefferson had an English translation of Stevin's book and was so impressed by his argument that he advocated the use of the decimal system in American money.

D. Galileo is also credited with a thought experiment related to falling objects: What if we drop two balls connected by some kind of bar? They now weigh at least twice as much and should fall twice as fast. What if the bar is very thin? Do the two balls constitute one object? What if a piece of string or a thin thread connects the two balls? Does that make them one object or two?

1. This thought experiment illustrates that the rate at which something falls cannot be dependent on its weight.
2. This idea is often credited to Galileo, but in fact it goes back to the 16th century and the work of Giovanni Benedetti.

IV. Another question Galileo explored was whether the velocity of a falling object—which increases as the object falls—increases as a function of time or a function of distance.

A. Velocity can be expressed either as a function of time or as a function of distance, but which expression is easier? Scientists up until the early 1600s debated this question.

B. Galileo eventually concluded that velocity increases at the same rate over each interval of time. We know that Galileo conducted experiments with balls rolling down inclined planes, showing that in each unit of time, the ball picks up the same increase in velocity. Thus, velocity is most easily expressed as a function of time.

C. With the understanding that velocity increases by the same amount for each unit of time, we might ask: How far does a dropped ball fall over a given period of time? To answer this question, Galileo represented the velocity at each interval of time by a small vertical line, a representation that would serve as an important precursor to calculus.

1. If a ball is dropped, starting with velocity zero, and we align the vertical lines that designate the successive velocities, they form a triangle.
2. Galileo realized that the distance the object would travel is precisely the area of this triangle: the base (the time that has elapsed) times half the height (the final velocity). In other words, the distance traveled under uniform acceleration is the elapsed time multiplied by half of the final velocity.
3. The idea of representing the velocities as vertical lines was a precursor to the development of analytic geometry.

V. Galileo also explored the vector decomposition of velocities.

A. Think again of throwing a ball. Galileo broke the velocity of the ball down into two parts, a horizontal component and a vertical component. He assumed that the horizontal component of the velocity does not change much, but the vertical component does.

B. We can treat the vertical component of the velocity as if we're working with a falling object. The vertical motion is initially upward or positive, but it decreases by the same amount in each unit of time.

C. Using this idea of decomposition, Galileo was the first person to show that the trajectory of a thrown object must be a parabola.

D. Galileo's idea of decomposing a velocity into two orthogonal components (two parts at right angles to each other) was foreshadowed by the work of Simon Stevin in decomposing forces. Stevin wrote a work called *Elements of the Art of Weighing*, in which he studied objects on an inclined plane.

1. Imagine that we have a slanted plane (with no friction on its surface) with an object sitting on it. The force of gravity is pulling this object down. How much force do we need to apply in order to keep the object stationary on the plane?
2. The actual force acting on the object is gravity, which is straight down, but the object can only move along the slanted plane.

3. Stevin realized that the amount of force needed to counteract the tendency of the object to slide could be determined by decomposing the vertical force into two parts, one force that ran parallel to the plane and another force at right angles to the plane.
4. We construct a rectangle so that one side is parallel to the plane and the vertical force (due to gravity) is the diagonal of the rectangle. The length of the side of the rectangle that is parallel to the plane describes the force needed to keep the object stationary.

VI. Galileo looked into the question of inertia, which he knew was also involved in explaining why we don't sense the motion of the Earth.

A. Initially, inertia was closely connected to the idea of impetus; it was thought to be a tendency of bodies to remain at rest. To start motion and continue motion, inertia must be overcome.

B. Galileo was the first scientist to see that inertia is a tendency of an object to keep traveling in the direction it has been traveling. If an object is in motion, it tends to stay in motion unless a force acts on it. If an object is stationary, it tends to remain stationary unless a force acts on it.

C. Galileo tried to use the concept of inertia to explain why we have no sense of movement even though we stand on an Earth that is spinning at an incredible speed.

1. Galileo considered "circular inertia"—the idea that if we are moving in a circle, we tend to continue this circular motion, and that is why we don't sense the motion of the Earth.
2. Later scientists would realize that circular inertia does not exist. Inertia only operates along straight lines.

D. It has been suggested that Galileo was not able to understand the tendency of objects to keep moving in a straight line at the same velocity forever because that idea assumes that space goes on forever, and Galileo was not prepared to accept the fact that space is infinite.

1. He still believed that the universe was enclosed within a sphere, and in fact, circles were important to his understanding of the universe.
2. Galileo's real contribution, however, was the recognition that an understanding of the world required mathematics. The universe, he said, "is written in the language of mathematics,

and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it."

VII. In our next lecture, we will continue with Pierre de Fermat and René Descartes, who transformed Galileo's idea of representing velocity as a sequence of vertical lines into analytic geometry and thus set the stage for the development of calculus.

Suggested Readings:

Cohen, *The Birth of a New Physics*, chaps. 4, 5.

Dijksterhuis, *The Mechanization of the World Picture*, 324–85.

Gindikin, *Tales of Mathematicians and Physicists*, 27–78.

Katz, *A History of Mathematics*, 314–21, 420–25.

Naess, *Galileo Galilei*.

Questions to Consider:

1. Galileo is universally considered to be an important figure in the development of science, but there is little agreement when it comes to pinning down his innovations. What do you see as his most significant contributions?
2. What *was* Galileo's contribution to our understanding of gravity?

Lecture Twelve

Fermat, Descartes, and Analytic Geometry

Scope: Pierre de Fermat, a lawyer from Toulouse, received his mathematical training in the algebraic tradition of François Viète. He read Latin translations of classical Greek works in mathematics and applied modern algebraic techniques to push this work forward. One of his earliest insights was a method for translating geometric problems into algebraic equations and vice versa, a discovery he shared with René Descartes. Fermat is the inventor of differential calculus, creating it specifically to solve the problem of finding the maximum value of a function. He discovered the rules for finding the area under the graph of $y = x^n$ for an arbitrary integer n. Fermat also did work with perfect numbers and Pythagorean triples, leaving his last theorem as a legacy to be solved by 20th-century mathematicians.

Outline

I. In this lecture, we explore the work of Pierre de Fermat (1601–1665) and René Descartes (1596–1650). We begin with Fermat, who was instrumental in the development of analytic geometry and calculus.

A. Fermat was born near Toulouse and studied mathematics in Bordeaux with some of the disciples of François Viète, a French algebraist from the 1500s. Fermat then went on to Orléans, where he earned a law degree before returning to Toulouse. He secured a position as a counselor to the Parliament and rose quickly in government service.

B. Fermat viewed mathematics as an avocation and published little during his lifetime. His son published his manuscripts after his death.

C. Fermat's work with mathematics began in the 1620s, when he learned of a lost book by Apollonius of Perga, the author of the *Conics*. This lost work, *Plane Loci*, dealt with two-dimensional curves. Fermat took it upon himself to prove some of Apollonius's results.

D. Fermat invented an entirely new way of approaching geometric problems that is now called *analytic geometry*. He interpreted geometric statements algebraically, though the correspondence can also be used to create geometric representations of algebraic expressions.

1. When we graph an algebraic expression, we use a horizontal axis and a vertical axis. We then plot points for each pair of values (x, y) that satisfies the equation.
2. As we plot the points on the *Cartesian coordinate axis*, the algebraic expression is transformed into a geometric curve.
3. Fermat translated the geometric statements of Apollonius into algebraic statements; he then used advanced algebra to find simple proofs of the geometric theorems.

E. Fermat constructed his graphs without a vertical axis. He represented one of the variables on a horizontal axis and then simply marked off the distance to the other variable, similar to what Galileo had done in representing velocity—but Fermat used points rather than Galileo's vertical lines.

F. In 1637, Fermat published his idea in *Introduction to Plane and Solid Loci*.

1. In the same year, René Descartes independently came up with the same idea for translating between geometric curves and algebraic expressions. Descartes published his work, *Geometry*, in a more comprehensive treatise called *Discourse on the Method for Rightly Directing One's Reason and Searching for Truth in the Sciences*.
2. Descartes' explanation of how to translate between geometric and algebraic expressions became much more popular than Fermat's, partly because Descartes used more efficient algebraic notation.

II. Descartes and Fermat differed in their approach to moving between geometry and algebra.

A. Descartes, like Fermat, started with the idea of interpreting a geometric object algebraically and then looked for geometric conclusions by studying the algebra.

B. Fermat, however, realized that he could get powerful results by moving in the other direction. He could graph an algebraic equation, such as an equation that describes a quadratic polynomial, and work with the resulting parabola instead of the

algebraic expression—perhaps seeing ideas that were not readily apparent in looking at the algebra alone.

1. Think, for example, of a parabola that opens downward. Looking at this geometrically, we can pose the question: What is the area between the horizontal axis and the parabola?
2. Again, looking at an expression geometrically, we can also see where an algebraic equation hits its largest or smallest value within some range of values.
3. The ability to translate an algebraic expression into a geometric representation led directly to the development of much of calculus.

III. As mentioned in an earlier lecture, Descartes coined the term *imaginary number* for the square root of a negative number.

A. In the 17th century, mathematicians were still unsure about the legitimacy of negative numbers. Earlier Chinese, Indian, and Islamic mathematicians had been divided on this issue.

1. Descartes was one of the last western European scientists to consider negative numbers to be less legitimate than positive numbers. He viewed the negative root of a polynomial (a negative solution to a polynomial equation) as a "false root."
2. By the middle of the 17th century, negative numbers were accepted as fully legitimate solutions to problems.

B. Descartes' work *Principles of Philosophy* (1644) set science back in some ways.

1. In four volumes, Descartes sought to lay out the principles on which all knowledge of the universe could be based. His foundations would replace those originally established by Aristotle.
2. Descartes stated quite forcefully that there could be no action without an actor, no action at a distance. If something is moving, it is doing so because something else is moving it.
3. He explained the motion of the planets by postulating a swirling ether that pervaded the universe and moved the planets with its own motion. People on Earth don't sense the motion because we are also embedded in the ether and moving with it.
4. When Isaac Newton published his *Principia* in the later 17th century, he was unable to explain how gravity could act at a distance without an intermediary. Many people criticized

Newton because he could not come up with a Cartesian-type mechanism to explain the action of gravity.

IV. Although Fermat published little, he was an active correspondent in mathematics; in particular, he exchanged letters with Father Marin Mersenne (1588–1648), a Catholic priest in Paris.

A. Mersenne was a key figure in the exchange of information among scientists of the time. He was a great friend of Descartes' and Christiaan Huygens's and a supporter of Galileo.

B. In 1636, Fermat tackled the problem of finding the area underneath a curve described by an algebraic expression, using what would come to be understood as integral calculus. Fermat used the method of exhaustion discovered by Eudoxus of Cnidus. In the same year, another mathematician, Gilles de Roberval, found the same formula as Fermat for the area under a curve that corresponds to an arbitrary polynomial.

C. In 1639, Fermat solved an important problem: Given an algebraic expression, how can its maximum (its greatest value) or its minimum (its least value) be found?

1. The idea is to represent the algebraic expression as a curve and then look at the line that is tangent to the curve at each point.
2. The slope of the tangent will be positive if the expression is increasing and negative if the expression is decreasing. For an expression that is at its highest or lowest value, the tangent line will have a slope of zero.
3. Fermat then looked at the problem of determining the slope of the tangent line at a single point. He considered two points that are very close to the point in question on the curve. It's possible to construct the equation of the straight line that goes through those two points, and the slope of that line is the rise over the run, or the change in the y value divided by the change in the x value.
4. Fermat saw that as the change in the x value gets smaller, the change in the y value also gets smaller. This ratio, in which both the numerator and denominator get closer to zero, will approach the slope of the tangent line at that particular point.
5. When we want to find the maximum or minimum value, we follow this procedure and look for those points where the slope of the tangent line is exactly equal to zero.

6. The fact that the slope of the tangent line is zero does not guarantee that we have found a maximum or a minimum value. But for most algebraic expressions, there are very few places where the function has a horizontal tangent line; thus, the number of places to look for the greatest value or the least value is limited.

D. Fermat found simple computational formulas for determining the slope of the tangent line, today called the *derivative* of the function. These formulas included rules for the derivative of any polynomial and for the derivative of an arbitrary rational power of the variable.

V. Fermat was also interested in perfect numbers and Pythagorean triples.

A. A perfect number is one that is equal to the sum of its proper divisors. For example, the sum of the proper divisors of 6—1, 2, and 3—is 6. Perfect numbers were important to the Greeks and later thinkers. Four perfect numbers were known to the Greeks: 6, 28, 496, and 8128.

B. All even perfect numbers are constructed by first finding a prime that is 1 less than a power of 2. For example, 6 is 2×3, and 3 is 1 less than a power of 2.

C. Fermat did important work in understanding when a number that is 1 less than a power of 2 can be a prime.

1. First, the exponent must be a prime: $2^8 - 1$ cannot be prime; $2^9 - 1$ cannot be prime; $2^{11} - 1$ might be prime. In fact, it's not ($2^{11} - 1 = 23 \times 89$), but it raises an interesting point.
2. The exponent in the last example is 11. Both 23 and 89 are 1 more than a multiple of 11. Fermat realized that if we raise 2 to a prime power and subtract 1, the only numbers that can divide into the result are 1 more than a multiple of the prime exponent.
3. This discovery would come to be known as *Fermat's little theorem*. It is the basis of the modern RSA public key cryptosystem that is used for secure communication over the Internet.
4. The problem of finding prime numbers equal to 1 less than a power of 2 would continue to interest mathematicians. These numbers are called *Mersenne primes* in honor of Father

Mersenne; the largest prime known today is of this form: $2^{32,582,657} - 1$.

D. In looking at Pythagorean triples (integers such as 3, 4, and 5, where $3^2 + 4^2 = 5^2$), Fermat wondered if he could find three positive integers that met the same condition when raised to a higher power (for example, $a^3 + b^3 = c^3$).

1. In the margins of his copy of *Arithmetica* by Diophantus, Fermat wrote: "It is impossible … in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvelous demonstration which this margin is too narrow to contain."
2. Fermat almost certainly did not have a proof of this result, which came to be known as *Fermat's last theorem*. This assertion, in fact, would not be settled until the end of the 20th century.

Suggested Readings:

Edwards, *Fermat's Last Theorem*, chap. 1.

Gindikin, *Tales of Mathematicians and Physicists*, 129–50.

Katz, *A History of Mathematics*, 432–42, 448–60, 470–73, 481–85.

Van der Waerden, *A History of Algebra*, chap. 3.

Questions to Consider:

1. Certain ideas seem to wait until the time is ripe for discovery, and then several people hit on them simultaneously. This is the case with analytic geometry. What were the pieces that needed to be in place before this means of translating between algebra and geometry could be discovered?
2. Richard Hamming has commented that he finds it intriguing that the simplest ideas in algebra—linear expressions, then quadratic expressions—correspond to the simplest ideas in geometry—straight lines, then conic sections. Is this surprising? Should it be?

Lecture Thirteen

Newton—Modeling the Universe

Scope: Isaac Newton applied his genius to astronomy, mechanics, optics, chemistry, and theology, as well as mathematics. His *Mathematical Principles of Natural Philosophy* solved the greatest scientific problem of his age: how it is that the Earth moves, yet we are unaware of its motion. Also in this book, known as the *Principia*, we find the first clear statement of the relationship between force and acceleration, as well as the first sophisticated use of calculus as a tool for modeling. In much of his work, Newton built on the insights of others, including John Wallis, Christiaan Huygens, Isaac Barrow, and James Gregory. Newton was a complex and difficult man, yet his *Principia* set the paradigm for how science should be done: First discover the mathematical essence of the phenomenon, then build a succession of ever-more complex and accurate mathematical models of reality.

Outline

I. Galileo had started the process of pulling together the five important threads of mathematics: astronomy, algebra, geometry, mechanics, and the mathematics of motion. Throughout the 17th century, various scientists would move this project forward, and at the end of the century, Isaac Newton (1642–1727) would weave these threads together into a masterpiece that solved the problem of celestial motion.

A. Newton's masterpiece is the *Mathematical Principles of Natural Philosophy*, in which he proposed to lay out the mathematical foundations of the natural world. The book is often simply referred to as the *Principia*.

B. When Newton began the *Principia* in 1684, he was trying to answer a question posed by Edmond Halley: Why do the planets move in elliptical orbits? Newton wrote up an explanation of earlier work he had done on the question, but he realized that a more complete treatment of the relationship between science and mathematics was needed; thus, he began work on the *Principia*, which was completed in 1687.

C. The Royal Society in London was unable to publish the *Principia*, as it had expended its funds on an expensive history of fishes that had not yet yielded any income. Fortunately, Halley personally put up the money to publish Newton's work.

D. The *Principia* begins with a clear explanation of inertia as we currently understand it, that is, the tendency of a body at rest to stay at rest and the tendency of a body moving at a uniform velocity to maintain that uniform velocity.

1. We also find in this book the first clear statement of the relationship between force and acceleration (the rate at which velocity changes): Force is directly proportional to acceleration. This relationship would be foundational to Newton's understanding of celestial mechanics.
2. In Newton's *Principia*, we find the first sophisticated use of calculus, although the modern terminology and notation of calculus would be developed by Leibniz.
3. The *Principia* is couched in the language of Euclidian geometry. It draws heavily on algebra as well as the mechanics of inertia and force, the mathematics of motion, and astronomy.

II. Newton is famous for having said, "If I have seen a little further, it is by standing on the shoulders of giants." It is certainly true that he built on the work of many earlier scientists and mathematicians, including Napier, Galileo, Fermat, Descartes, and others.

A. One of the mathematicians from whom Newton learned algebra was John Wallis (1616–1703), a code breaker who worked for the Parliamentarians during the English Civil War. Wallis read William Oughtred's *Clavis Mathematicae* (*Mathematical Key*, 1631) and became fascinated with the power of algebra. Two years after his introduction to algebra, he became a professor of mathematics at Oxford.

1. In 1655, Wallis published *Arithmetica Infinitorum* (*Arithmetic of the Infinities*), which would have a great influence on Newton.
2. The book has general methods for finding the slopes of curves and areas underneath curves. It also has wonderful formulas, including a formula for π.

3. Wallis realized that $\frac{\pi}{2}$ could be written as an infinite product: $\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\cdots$. For each fraction, either the numerator or the denominator is increased by 2. As more terms are taken in this product, it approaches the value of $\frac{\pi}{2}$.
4. Wallis also invented the symbol for infinity: ∞.

B. Another of the giants on whose shoulders Newton stood was Dutch scientist Christiaan Huygens (1629–1695). His father, Constantine, was a scientist, diplomat, philosopher, and a close friend of Descartes.

1. Huygens's interests were wide ranging. In 1655, he became the first person to discover a Moon circling Saturn. In 1656, he invented the pendulum clock, figuring out the mathematics and mechanics of a swinging pendulum. Huygens is also credited with designing an internal combustion engine to run on gunpowder. In 1675, he patented a pocket watch.
2. In 1661, Huygens met Wallis, and that year marked the beginning of his work in mathematics and physics. Huygens understood the concept of inertia, although he never stated it as clearly as Newton eventually would.
3. Huygens is often credited with the discovery of the inverse square law of gravitational attraction: The force exerted by gravity diminishes as the square of the distance when an object moves farther away from the attracting body. If we double the distance from the attracting body, the force decreases to a quarter of its previous value.
4. This law grew out of Huygens's exploration of the force necessary to keep a rock in its orbit as it whirls on a string. If you were to tie a rock to a string and whirl it around your head, you would feel a force pulling on the string. This is the effect of inertia: the tendency of the rock to travel in a straight line. You must constantly pull on the string to keep the rock whirling around your head.
5. In fact, the famous story of Newton and the falling apple is probably true; it was recorded by Newton's niece.
   a. According to the story, when Newton saw the apple fall, he began to think about gravity and wondered how far the force of gravity extends.

b. If the force of gravity extends to the Moon, then as the Moon travels around the Earth, it is constantly pulled toward the Earth. At the same time, the Moon has a tendency to move along a tangent line. Combining these motions yields the actual motion of the Moon as it circles the Earth.

C. Newton's teacher in mathematics was Isaac Barrow, a professor of Greek at Cambridge University and a professor of mathematics at Gresham College in London. In 1663, Barrow became the first Lucasian Professor of Mathematics, a position held today by Stephen Hawking. In 1669, when Barrow became the Royal Chaplain, he passed the Lucasian professorship to Newton.

1. In 1670, Newton edited and published the *Geometrical Lectures*, a series of lectures given by Barrow. The lectures illustrate the connection between areas and tangents. They also begin to tie together the two aspects of calculus: differential calculus, which deals with slopes of tangent lines, and integral calculus, which deals with areas.
2. Barrow was beginning to understand the connection between these two aspects of calculus, a connection that today we refer to as the *fundamental theorem of calculus*. Newton understood this idea and was able to build on it effectively as he wrote the *Principia*.

D. Newton also drew on the work of the Scotsman James Gregory (1638–1675), who had studied with Stephano Angeli in Italy.

1. Gregory translated the problem of finding the length of a curve to one of finding the area underneath a different curve. This area then could be evaluated using the methods of integral calculus.
2. Gregory also discovered general methods for finding power series. A *power series* is a polynomial of infinite degree.
   a. Given a function, we consider a sequence of polynomials, each of higher degree than the previous one, that get closer to the function we want to model as the degree increases.
   b. In the limit, we say that this polynomial of infinite degree (power series) is equal to the function in question.

III. Newton had an unfortunate childhood. His father died before he was born; his mother remarried shortly thereafter and packed Newton off to

live with her parents, with whom he apparently did not get along. In later life, one of Newton's students, William Whiston, said, "Newton was of the most fearful, cautious and suspicious temper that I ever knew."

A. Newton published very little about calculus. The *Principia* uses calculus, but it is disguised in the language of Euclidean geometry. A few earlier manuscripts also include some of Newton's ideas about calculus, most of which were developed in the years 1665–1667, when Newton was forced to leave Cambridge and return home because of an outbreak of the plague.

1. Newton circulated some of his ideas in manuscripts in 1666, 1669, and 1691, but it's not correct to say that he invented calculus in the sense of establishing a foundation on which others could build. That job would be left to Gottfried Leibniz.
2. Later on, a controversy arose between Leibniz and Newton about who originated the ideas of calculus. The controversy ultimately involved mathematicians in England and on the Continent.

B. Newton spent far more time on chemistry than he did on physics or mathematics. He was also fascinated with theology and studied the Bible for numerical patterns.

C. Newton was elected to Parliament and eventually left his position at Cambridge. He later became Warden of the Royal Mint. Newton published his book on optics in 1704, an important work in which he explains the refraction of light and the idea of light as a particle. He was knighted in 1705.

D. Newton became president of the Royal Society and maintained that position until his death. He influenced the society to issue a statement that *he*, rather than Leibniz, had invented calculus.

E. Newton was a complex figure without any close family or associates. He devoted himself to mathematics, physics, and chemistry, yet he was reluctant to publish his findings. In the words of the economist John Maynard Keynes:

> Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago.

IV. In the next lecture, we will turn to Leibniz and the Bernoullis and the continuing story of calculus.

Suggested Readings:

Cohen, *The Birth of a New Physics*, chap. 7.

Edwards, *The Historical Development of the Calculus*, chap. 8.

Gindikin, *Tales of Mathematicians and Physicists*, 79–91.

Gleick, *Isaac Newton*.

Katz, *A History of Mathematics*, 496–522.

Questions to Consider:

1. Newton would not have considered himself to be a mathematician; he was a natural philosopher. Was it even possible to be a "mathematician" in the modern sense at that time?
2. It took a long time before scientists accepted Newton's explanation of gravity because it lacked any mechanism by which gravity could act. Newton's famous response to these critics was: "I frame no hypotheses." What did he mean? Is this a legitimate response?

Lecture Fourteen

Leibniz and the Emergence of Calculus

Scope: Gottfried Leibniz learned the fundamentals on which he would build calculus directly from Christiaan Huygens. Many of these basic ideas can be traced back to the work of earlier mathematicians, from Eudoxus of Cnidus, through Ibn al-Haytham and Bhaskara, to Fermat, Descartes, Wallace, Gregory, Barrow, and Newton. Unlike Newton, Leibniz's approach to calculus was based on sums and differences, and it was Leibniz who first published a full statement of the fundamental theorem of calculus, explaining in clear language the connection between differential calculus (the finding of slopes) and integral calculus (the finding of areas). Leibniz had two apt pupils who quickly became his collaborators: the Swiss brothers Jakob and Johann Bernoulli. The Bernoullis tackled the isochrone problem, established l'Hospital's rule, and did valuable work in probability.

Outline

I. Newton understood calculus in its full complexity, but he never explained it in a way that others could build on and use. That job would fall to Gottfried Leibniz (1646–1716) and the Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748).

II. Calculus is now one of the foundational fields of mathematics, lying at the heart of most of the mathematics that is done today.

- **A.** The critical idea behind integral calculus is that of finding the area underneath a curve by subdividing it into small rectangles and adding up their areas to arrive at an approximation of the area underneath the curve. As the rectangles get smaller, the approximation becomes more accurate. This is the method of exhaustion developed by Eudoxus of Cnidus and used so successfully by Archimedes.
- **B.** Not only the ancient Greeks but also Islamic and Chinese mathematicians used this method. Ibn al-Haytham used it to find the volume of the dome obtained by rotating a parabola.
- **C.** The other side of calculus is differential calculus, which looks at rates of change. For example, with motion, if we know the position

of an object at any given time, the velocity is the rate at which the position of that object changes. If we know the velocity, then the acceleration is the rate of change of the velocity. Geometrically, the rate of change can be thought of as the slope of the tangent line to a curve.

D. Again, the idea of building mathematical models around rates of change can be traced back to the Indian astronomers of A.D. 400–500, who used the rate at which the sine function changes to find interpolating formulas. This application was nearly perfected in the 12th century by Bhaskara, who in some sense invented the derivative.

E. In the 17th century, Fermat and Descartes invented analytic geometry, the translation of a geometric object into its algebraic representation and vice versa. Fermat found many of the formulas commonly associated with calculus, such as those for finding the area underneath a polynomial curve or the slope of the tangent to an arbitrary polynomial.

F. Wallis and Gregory furthered these ideas. They came up with the idea of power series, infinite polynomials that Newton would use in writing the *Principia* and that would become essential to the development of calculus in the 18th century. Later, we'll see how Leonhard Euler used power series to exploit the full potential of calculus.

G. We've also seen how Isaac Barrow, in his *Geometrical Lectures*, tied together differential calculus and integral calculus. The problems associated with these two parts of calculus—figuring out rates of change and finding the area under curves—are related.

1. If we know the slope of a tangent line to a curve at every point (we know its derivative), we can find an expression for that curve by using the idea of integration.
2. This process also works the other way. If we know the area underneath a curve from, say, 0 to any point x (we know its integral), then we can use differential calculus to find an expression for the curve.
3. Thus, integration and differentiation are reverse processes of each other. This realization would become known as the *fundamental theorem of calculus* and is the origin of the name *calculus* (a tool for calculating) for this branch of mathematics.

III. Newton was interested in calculus to explain motion, particularly the motion of the planets.

A. Newton used calculus to understand the relationship among the position of an object, such as a planet; its velocity; and its acceleration.

B. The fundamental law of gravitational attraction (the force is inversely proportional to the square of the distance) yields what is called a *differential equation*, an equation that connects position, velocity, and acceleration.

1. Force, which is proportional to acceleration, which is also the derivative of velocity, is equal to a constant divided by the square of the distance, which is determined by the position.
2. Thus, position determines the force, which determines the rate at which velocity changes, and the velocity determines the rate at which the position changes.
3. Calculus enables us to disentangle these and find exact expressions for position, velocity, and acceleration.

IV. Gottfried Leibniz was introduced to advanced mathematics by Huygens.

A. Leibniz was born in Leipzig. He studied philosophy and law and became a private secretary, librarian, and personal advisor to a nobleman in Frankfurt. He was also interested in science, and partly because he had a patron, he was able to travel to pursue his studies.

B. In 1672, Leibniz went to Paris, where he met Christiaan Huygens. Huygens took Leibniz under his wing and introduced him to the work of Wallis and others who were involved in the development of calculus. Two years later, Leibniz traveled to London and was elected as a fellow in the Royal Society.

C. Leibniz learned that Newton was also working with calculus and wrote to him, asking about his results. In a guarded response, Newton shared some of his results. When Leibniz wrote back, hoping to further the relationship, Newton cut off the correspondence. This exchange would be the basis for Newton's later claim that Leibniz had stolen his ideas.

D. In 1676, Leibniz returned to Germany and became the librarian to the Duke of Hanover. In this position, he devoted himself to the study of calculus. Unlike Newton, Leibniz began to publish his results, which first appeared in 1684 in the earliest scientific journal in Germany, the *Acta Eruditorum*.

E. In 1686, Leibniz published his full statement of the fundamental theorem of calculus, explaining in clear language the connection between differential calculus (the finding of slopes) and integral calculus (the finding of areas).

F. The controversy between Newton and Leibniz did not break out until 1711. It would damage relations between English and Continental mathematicians for over a century.

V. As we said, Newton was interested in calculus as the mathematics of motion. He wrote about what we today call derivatives as "fluxions." Leibniz, however, had a different way of thinking about calculus.

A. Leibniz looked at integral calculus as not necessarily related to areas but as limits of sums of products.

1. Each of the rectangles used in the approximation of an area has a small width that we denote by Δx. If the height is given by y, then the area of each rectangle is given by the product $y\Delta x$.
2. The sum of these products is the approximation to the area. The integral is evaluated by looking at what happens to this sum of products as Δx becomes "infinitesimally small."

B. Leibniz studied rates of change, not as velocities and accelerations, but as ratios of differences.

1. We are interested in the change in y (Δy) as x changes.
2. Leibniz wrote this rate of change as the ratio $\frac{\Delta y}{\Delta x}$.
3. As Δx gets smaller, so does Δy. The derivative is the ratio that is achieved once both quantities are "infinitesimally small."

VI. Two other mathematicians who played a critical role at the end of the 17th century and into the early 18th century were Jakob and Johann Bernoulli.

A. Jakob started his career by studying theology at the University of Basel, but while traveling in France, he began to study mechanics and mathematics. He spent time in the Netherlands and England before returning to Basel in 1683 and later taking the position of chair of mathematics at the university.

B. Johann entered the University of Basel in 1683 with the intention of studying medicine. Under the influence of his older brother, however, he became interested in mathematics.

C. In 1687, the two brothers discovered the papers that Leibniz had begun to publish, and in 1690, Jakob published his first important work on calculus, the solution of the isochrone problem.

1. The isochrone is the shape of a ramp designed so that wherever a ball is placed, it will take exactly the same amount of time to reach the end.
2. Jakob Bernoulli used the tools of calculus to determine the shape of the isochrone. It is the curve known as the *cycloid*.

D. Johann traveled to Geneva and Paris. In 1691, he began a correspondence with Leibniz, and the two collaborated over the succeeding years in building up the foundations of calculus.

E. Johann was sponsored in Paris by a nobleman named Guillaume de l'Hospital. In exchange for this sponsorship, Johann agreed that l'Hospital was free to use any of Bernoulli's mathematical results as his own.

1. In 1696, l'Hospital published the first textbook on calculus, *Analyse des Infiniments Petits* (*Analysis of the Infinitely Small*).
2. Contained in this textbook is a result that is today called *l'Hospital's rule*. This is a general method that uses derivatives to determine what happens to the ratio of two quantities as both approach zero.

F. In 1705, Jakob Bernoulli died, and Johann took over his chair at the University of Basel. Over the next several years, Johann completed the work that his brother had begun on the mathematics of probability, and he published this work as the *Ars Conjectandi* (*Art of Conjecture*, 1713).

1. One of the important ideas in this book is the *Bernoulli polynomial*, a polynomial that is efficient for finding the sums of powers. As mentioned earlier, al-Haytham had found a formula for the sum of the fourth powers, and other methods exist for finding the sums of consecutive powers.
2. Bernoulli polynomials would turn out to be the most efficient way of finding these sums. They also gave rise to an interesting sequence of rational numbers known as the

Bernoulli numbers: $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730} \cdots$. We will see later how these numbers are connected to Fermat's last theorem.

VII. In the next lecture, we'll turn to Leonhard Euler, a student of Johann Bernoulli and perhaps the greatest mathematician who ever lived.

Suggested Readings:

Edwards, *The Historical Development of the Calculus*, chap. 9.

Gindikin, *Tales of Mathematicians and Physicists*, 151–170.

Katz, *A History of Mathematics*, 522–35, 544–52, 596–601.

Questions to Consider:

1. Great scientists often have great egos. How does this both help and handicap them?
2. Leibniz was always suspicious of universities, believing that they isolated scholars of different disciplines from one another, hindering the cross-fertilization of ideas. Today, it is difficult to get a job as private librarian to a duke. How do modern scholars who wish to avoid the "silos of academe" accomplish this?

Lecture Fifteen

Euler—Calculus Proves Its Promise

Scope: Leonard Euler dominated 18th-century mathematics despite his failing eyesight and near total blindness by age 64. Taught by Johann Bernoulli, he spent his career in St. Petersburg and Berlin. He solved many practical problems in shipbuilding, navigation, astronomy, and ballistics and was a master expositor. His *Introduction to Analysis of the Infinite* (1748) became one of the most influential books in mathematics. Euler was also the master of the power series and began the process of understanding complex numbers as legitimate numbers representing positions in a two-dimensional plane. He greatly advanced our understanding of number theory and made the first significant progress on Fermat's last theorem.

Outline

I. Leonhard Euler (1707–1783) dominated the field of mathematics in the 18th century. By the time of his death, his contemporaries believed that he had solved most of the important problems.

A. Euler, the son of a preacher, was born near Basel. He attended the University of Basel with the intention of studying theology, but Johann Bernoulli saw his remarkable talent and encouraged him to pursue mathematics.

B. Euler spent most of his life in St. Petersburg and Berlin. He was incredibly prolific—his collected publications run to more than 25,000 pages—and he worked in all areas of mathematics, including calculus, geometry, number theory, and mechanics.

C. Euler also did work in optics, astronomy, and fluid dynamics. He was interested in shipbuilding, navigation, ballistics, cartography, and the field of science education. In Berlin, he became director of the astronomical observatory. He also worked for Frederick the Great, designing spectacles and water fountains for the emperor.

D. Euler published 25 expository books on mathematics, science, philosophy, and music. One of the most popular books of the 18th century was Euler's *Letters to a German Princess*, in which he explained to a general audience basic ideas of science and astronomy.

E. For the last 12 years of his life, Euler was completely blind, yet his output never slowed. At his death, he still had 3000 pages of scientific articles waiting to be published. One scholar has estimated that in the period from 1725 to 1800, Euler accounted for one-third of all the published research in mathematics, mathematical physics, and engineering mechanics.

F. Some of Euler's most important mathematical work was in applying the ideas of calculus to dynamical systems, that is, systems that are in flux. To do this, he set up differential equations, equations that relate derivatives to the function under study. He also showed that the use of power series was essential to studying problems in dynamical systems.

II. One of the most important books on mathematics ever published was Euler's *Introduction to Analysis of the Infinite* (1748), which includes his explorations of the power series for the exponential function.

A. Recall from Lecture Ten that Napier advanced the idea of the exponent as a variable—for example, 2^x—in which x can be any real number.

1. In the 17th century, mathematicians explored that exponential function for which the base is the base of the natural logarithm.
2. If we raise this number, e (which is about 2.71), to the x power, we get a function whose derivative (the slope of the tangent line at any point) is equal to e^x. This remarkable function has a derivative that is equal to itself.

B. In his *Analysis of the Infinite*, Euler discusses the relationship between the power series expansion of e^x and power series expansions of the trigonometric functions sine and cosine.

1. Recall that the sine is the half-chord. Consider the chord that connects the two endpoints of an arc of a circle. The sine is a function of half of that arc, and it returns the value of the half-chord. The value of the half-chord depends on the size of the original circle.
2. Instead of working with a radius designed to match a circumference of 360°, Euler decided to work with a radius of 1, which means that the circumference is 2π. This measure of arc length is called a *radian*. The complete circumference is 2π radians.
3. An arc that is a quarter of a circle will be $\frac{1}{4}(2\pi)$, or $\frac{\pi}{2}$ radians. An arc that is an eighth of a circle (45°) will be $\frac{\pi}{4}$ radians.
4. Using power series, Euler showed how to take an exponential with an imaginary number in the exponent: e^{ix}; $i=\sqrt{-1}$. This is equal to a complex number in which the real part is $\cos x$ and the imaginary part is $i\sin x$: $e^{ix}=\cos x+i\sin x$.

III. Euler now began the process of understanding complex numbers as representing points in a two-dimensional plane, with zero located at the center where the horizontal and vertical axes cross.

A. For a complex number, such as $2+3i$, the horizontal axis is used to mark the real number, and the vertical axis marks the imaginary number.

1. The point that corresponds to $2+3i$ is found by going two units to the right along the horizontal axis and three units up, parallel to the imaginary axis.
2. Thus, every complex number represents some point in this complex plane. We are simply extending the idea of numbers along a real line to numbers that exist in the complex plane.

B. What if we raise e to a complex power? Let's consider e^{2+3i}.

1. Recall how we deal with a sum in the exponent: $2^{4+7}=2^4\times 2^7$. Thus, $e^{2+3i}=e^2\times e^{3i}$.
2. By Euler's formula, e^{3i} equals $\cos 3+i\sin 3$. If we square both the real part and the imaginary part of that number, we get $\cos^2 3+\sin^2 3=1$.
3. No matter what value for x we use, we always have that $\cos^2 x+\sin^2 x=1$. Thus, e^{ix} (in this case, e^{3i}) will always be a point that lies exactly 1 unit away from the origin.
4. To find the point that corresponds to $e^{2+3i}=e^2\cos 3+ie^2\sin 3$, we take the ray from the origin through the point at e^{3i}, and move distance e^2 out from the origin.

5. The point in the complex plane that is exactly e units away from the origin and is at an angle of 45° above the horizontal line (45° corresponding to an arc length of $\frac{\pi}{4}$ *radians*), is denoted by $e^{1+\frac{\pi}{4}i}$.
6. As we will see in upcoming lectures, this understanding of the complex plane in terms of exponentials of complex numbers is foundational to much of modern mathematics.

IV. In 1726, at the age of 19, Euler published his first paper, related to the isochrone problem solved by the Bernoullis.

A. Euler added a resisting medium, such as water or air, to the problem. That would change the shape of the ramp on which the ball could be placed at any position and still reach the end in the same time. This was the beginning of Euler's work in fluid dynamics.

B. A few years later, two of Johann Bernoulli's sons, Daniel and Nicholas, were invited by Peter the Great to set up an academy of science in St. Petersburg. When Nicholas died suddenly, Daniel convinced the Russian court to invite Euler to the academy. There, the two explored dynamical systems, such as springs and vibrating strings and drumheads, as well as fluid dynamics.

C. In 1733, Euler took over Daniel's position as chair of mathematics at the institute and also married Katharina Gsell, the daughter of a Swiss painter working for the Russian court.

D. In 1735, Euler made his reputation as a mathematician by solving one of the great problems of the time, summing the reciprocals of the squares. The Bernoullis had shown that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}\cdots$ approaches a fixed value, but they didn't know what that value was. Euler showed that this infinite summation approaches the quantity $\frac{\pi^2}{6}$.

E. In 1738, Euler lost the sight in one of his eyes as the result of an infection. He lost the sight in his remaining eye in 1771 after unsuccessful cataract surgery.

F. In 1741, Euler was invited by Frederick the Great to fill the position of director of the astronomical observatory in Berlin. There, he met the great French philosophers of the Enlightenment but had little time for them. Frederick sought brilliant conversationalists and was disappointed in his mathematician. Euler was never happy in Berlin.

G. In 1766, Euler returned to St. Petersburg, where he would spend the rest of his life. He died in 1783 of a brain hemorrhage.

V. Among Euler's many mathematical accomplishments are his studies of the harmonic series and of Fermat's work in number theory.

A. The harmonic series is the sum of the reciprocals of the integers: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots$. This sum does not approach any finite number—it gets arbitrarily large as we add more terms—but Euler realized that it grows in the same way as the logarithm of the largest number whose reciprocal we take.

1. If we take this sum of reciprocals up to $\frac{1}{n}$, it is about equal to the natural logarithm of n.
2. Euler went beyond this, however. He was interested in finding the size of the difference between the harmonic series and the natural logarithm, and he was able to show that it is about 0.577.
3. Today, we refer to this number as gamma (γ); this constant is usually called *Euler's gamma*.

B. Euler was also interested in Fermat's work in number theory and was able to prove or find counterexamples to many of Fermat's statements. The first case of Fermat's last theorem is $x^3+y^3=z^3$. Euler published a proof that it is impossible to find three positive integers that satisfy this equation.

VI. We close with a brief look at some of the other mathematicians of the 18th century.

A. Jean le Rond d'Alembert (1717–1783) worked with Euler on many projects and explored some of the same problems in dynamical systems.

B. Another great mathematician of this era was the Italian Joseph-Louis Lagrange (1736–1813). He succeeded to Euler's position at the science academy in Berlin when Euler returned to St. Petersburg. Later, Lagrange became one of the first professors at the world's first institute of technology, the École Polytechnique in Paris.

C. A female mathematician, Maria Agnesi (1718–1799), wrote the first calculus text in Italian. Her book was so insightful that it was translated into many languages and widely used. In 1750, she was offered the chair of mathematics at the University of Bologna, but she declined it.

VII. By the end of the 18th century, many believed that Euler had solved all of the important problems in mathematics, but as we will see in the next lectures, entirely new areas of mathematics would emerge in the 19th century.

Suggested Readings:

Dunham, *Euler: The Master of Us All.*

Gindikin, *Tales of Mathematicians and Physicists*, 171–212.

James, *Remarkable Mathematicians*, chaps. on Euler, d'Alembert, and Lagrange.

Katz, *A History of Mathematics*, 544–90, 601–5, 610–21.

Questions to Consider:

1. Is it mere coincidence that so many great mathematicians came from Basel in the late 17th and early 18th centuries?
2. While the mathematics that Euler discovered would almost certainly have been found eventually, the fact that he lived greatly accelerated the development of science and mathematics. For whom else from the history of science could that claim be made?

Lecture Sixteen

Geometry—From Alhambra to Escher

Scope: In this lecture, we'll look at the interplay between art and mathematics, specifically the study of symmetrical transformations. We see many different kinds of symmetry in art, including mirror symmetry, rotational symmetry, and dilations. The artist Maurits Escher made particular use of modular transformations, which involve translating a point in the complex plane to another point that is the negative of the reciprocal of the first. We'll also look at Möbius transformations and a way of understanding these transformations called *projection*. Most people are familiar with the Mercator projection, our way of representing the spherical surface of the Earth on a flat map, but the idea of the projective plane may be new. This surface requires us to think in four-dimensional space.

Outline

I. In this lecture, we will see how art has influenced mathematics, particularly geometry, over the past few centuries.

A. The Lion Court at Alhambra in Granada is an excellent example of symmetry in art. For something to be symmetric means that it is left unchanged under some kind of a transformation. A mirror reflection of the Lion Court would be unchanged from the original.

B. The lion basin at the center of the court has a different kind of symmetry. Looking down on the basin from above, we see that it is surrounded by 12 lions equally spaced around the circumference of the basin. Thus, if we rotate the basin by 30°, the configuration is unchanged. The term for this configuration is *12-fold rotational symmetry*.

C. Translational symmetry appears throughout the palace at Alhambra. One example is when a pattern, often called a *wallpaper pattern*, repeats both horizontally and vertically. An arbitrarily large surface can be tiled in a pattern that has this translational symmetry because the pattern continues in all directions forever.

D. We see other examples of 3-fold, 4-fold, and 6-fold rotational symmetry in the patterns at Alhambra. Note that some of the colors or depictions of the figures seem to intentionally disrupt the symmetry. This breaking of the symmetry invites the observer to look more closely at the patterns.

E. We see another example of a wallpaper pattern from a metal door at Alcazar, the castle in Seville. This pattern has both 4-fold rotational and mirror symmetries, but in this case, the rotational symmetry has its center in the circles on the door, while the center of the mirror symmetry is at a different point.

F. Another important kind of symmetry is called a *dilation*. We see this in images in which progressively smaller copies of the entire picture are nested within it so that the smaller copies seem to continue into infinity. If we expand one of the small pictures, we recover the original picture. A dilation can be either an expansion or a contraction.

II. The graphic artist Maurits Escher (1898–1972) traveled to Alhambra in 1922. He was intrigued by its art and returned in 1936 to study the symmetry of the patterns used in the tiling of the walls. When he returned to the Netherlands, he began to explore the mathematics behind these patterns.

A. In 1924, George Pólya, one of the great Hungarian mathematicians of the 20th century, published a paper describing the 17 different types of wallpaper patterns that incorporated various rotational or mirror symmetries. This study raised some interesting questions.

1. What if, instead of a translational pattern in two dimensions, we think of a translational pattern in three dimensions, one that will be unchanged as we translate horizontally, vertically, or in the third dimension of depth? How many ways can we combine such an invariance with various rotational and mirror symmetries?
2. In fact, there are 320 different ways of combining three-dimensional translation invariance with other symmetries.

B. Escher began to create drawings that used all the wallpaper patterns; he then explored other transformations that mathematicians had studied, trying to create art that was unchanged under these transformations.

C. Modular transformations were one group that Escher studied. The complex numbers and complex plane that we discussed in the last lecture are critical to understanding modular transformations.

1. There are two basic modular transformations. In the first of these, an object is translated by one or more units to the right or left.
2. We now have a translational symmetry—but only in the horizontal direction, not a vertical translational symmetry.
3. The second symmetry that comes into play in modular transformations is a kind of inversion. We take a point in the complex plane (z) and translate it to the point that is the negative of the reciprocal of z ($-\frac{1}{z}$).
4. The modular transformations produce a pattern that reduces in size as it gets closer to the horizontal axis. Escher wrapped this pattern into a circle to produce his print *Circle Limit III*.

III. Another type of transformation is credited to August Möbius (1790–1868), an astronomer from Leipzig who had studied with Carl Friedrich Gauss.

A. The name Möbius is usually associated with the Möbius band, a strip of paper twisted so that it has only one side.

B. The group of transformations that Möbius studied includes translations, rotations, dilations, and simple reciprocals. A short video clip created by Doug Arnold and Jonathon Rogness at the University of Minnesota shows these Möbius transformations and how they can be combined.

C. The video clip also shows how to simplify these transformations by using *projection*.

1. We consider a sphere that sits above the plane and a region of the sphere that is projected down onto the plane.
2. We can make translations by moving the sphere around the plane, dilations by moving the sphere up and down, and rotations by rotating the sphere around a vertical axis. Inversion becomes a simple operation of rotating the sphere around a horizontal axis.
3. We can now combine these Möbius transformations just by combining motions of the sphere. We see, for example, a rotation of the sphere around the horizontal axis, followed by a

rotation of the sphere around a vertical axis, followed by a translation of the sphere by moving it.

4. The idea of projection involves projecting a region or collection of points that lies on one surface onto another surface.
5. Projections will come to play an important role in mathematics, and they can be traced back to the understanding of perspective that was developed by the artists of the Renaissance.

IV. Albrecht Dürer (1471–1528) was a Renaissance artist and mathematician.

A. Dürer was born in Nuremberg and studied both art and mathematics in Italy. When he returned to Germany, he wrote the first book of mathematics in German.

B. Among Dürer's many books on mathematics was one on compass and straightedge constructions. His famous *Adam and Eve* (1504) used compass and straightedge constructions to work out the geometric proportions of his figures.

C. Dürer's *St. Jerome* (1514) is an excellent example of the use of perspective. Note in particular how shapes change depending on the perspective. The bottom of the pot hanging from the ceiling in this picture is a circle, but when viewed from the side, in perspective, it appears to be an ellipse. This aspect of perspective would be studied extensively in the 19th century.

V. We know projections from a number of sources, including the *Mercator projection*, which is a common way of representing the surface of the Earth. This projection was developed by Gerardus Mercator (1512–1594).

A. The idea is to represent the spherical Earth on a flat map. We wrap a cylinder around the Earth, then project out from the axis through the North and South Poles through the surface to the Earth onto the cylinder.

B. Once the surface of the Earth is represented on the cylinder, we simply open it up to get a flat, rectangular map.

VI. Another mathematician who did important work on projections was Jean-Victor Poncelet (1788–1867).

A. Poncelet was interested in the properties that are left unchanged in a projection. As we saw with perspective, a circle becomes an ellipse, but we can't find a perspective that changes a circle into a square. Poncelet asked: What properties are left unchanged by different kinds of projections?

B. In the course of his study of this question, Poncelet realized that there is a basic duality between points and lines.

1. If we have two points in the plane, there is a unique line that goes through them. If we have two lines in the plane, they will cross at a unique point, unless the two lines are parallel.
2. Poncelet decided to allow the parallel lines to meet in infinity. This leads to what is known as the *projective plane*.
3. The basic idea behind the projective plane is to replace each of the lines that radiates from the origin in three-dimensional space with a single point. To understand the resulting surface, we would need to think in four-dimensional space.

C. Geometers have been ingenious in trying to visualize the projective plane. Werner Boy came up with one representation, known as the *Boy surface*, in 1901.

Suggested Readings:

Gray, *Worlds Out of Nothing*, 1–78.

James, *Remarkable Mathematicians*, chaps. on Monge, Poncelet, and Pólya.

Katz, *A History of Mathematics*, 389–93, 460–62, 633–35, 785–89.

Schattschneider, *M. C. Escher*.

Questions to Consider:

1. Symmetry is one of the basic concepts of mathematics that both appeals to our aesthetic sense and often is a reliable guide to what to expect of the world around us. Is there a connection between the aesthetic appeal of symmetry and its prevalence in nature?
2. Should we be surprised that there is a strong connection between mathematics and visual art?

Lecture Seventeen

Gauss—Invention of Differential Geometry

Scope: Carl Friedrich Gauss was the last mathematician whose research spanned all of mathematics, and he also made important contributions to astronomy as well as electricity and magnetism. In 1801, he published his *Investigations of Arithmetic*, the foundation of modern number theory. By the 1820s, he was focused on the geometric problem of distances on spheres and other surfaces. His work would establish the basis for the non-Euclidean geometries of the 19th century. Gauss's interest in elliptic functions would begin to forge connections between geometry on the one hand and problems in calculus and the study of functions on the other.

Outline

I. Carl Friedrich Gauss (1777–1855) might be one challenger to Leonhard Euler for the title of "greatest mathematician."

- **A.** Gauss was born in Brunswick, Germany, the son of a laborer. The Duke of Brunswick became his patron and supported his early work in mathematics and science. Later, Gauss would become the director of the astronomical observatory in Göttingen and would eventually hold the chair of mathematics at the University of Göttingen.
- **B.** Gauss did groundbreaking work in a number of areas of mathematics, including algebra, geometry, and number theory, as well as important work in astronomy. He was also noted for his work in electricity and magnetism, and the unit called the *gauss* is named for him.
- **C.** In 1801, Gauss published *Investigations of Arithmetic*, a valuable work on number theory. One of the mathematicians who studied this book was Sophie Germain (1776–1831), who corresponded with many great mathematicians of her day under the pseudonym M. Le Blanc. We'll return to Germain later when we discuss the eventual proof of Fermat's last theorem.

II. In this lecture, we'll focus on Gauss's work in geometry. We begin by returning to Euclid's *Elements*.

- **A.** Euclid's *Elements* includes five postulates.
  1. Two points determine a line.
  2. The line through these points is unique.
  3. A center point and a radius length uniquely determine a circle.
  4. All right angles are equivalent.
  5. Given two lines and a line that cuts across them, if the interior angles on one side of the line that cuts across are less than right angles, then the original two lines must meet.
- **B.** The fifth postulate, known as the *parallel postulate*, can be restated as follows: Given a line and a point that is not on that line, there is exactly one line that passes through the point and never intersects the given line. In other words, there is exactly one parallel line through the point.
- **C.** Many people tried to prove that the parallel postulate follows from the other assumptions about geometry, and one who made progress on it was Girolamo Saccheri (1667–1733).
  1. Saccheri explored contrary cases of the postulate: What if there was no parallel line? What if there was more than one parallel line?
  2. Saccheri showed that if there is no parallel line, then the sum of the angles in any triangle must be greater than 180°. Further, if there is more than one parallel line, then the sum of the angles in any triangle must be less than 180°.
- **D.** The next person to make progress on the postulate was Johann Lambert (1728–1777). He was primarily concerned with the possibility that there is more than one parallel line.
  1. Lambert explored the geometry that would follow if there was more than one parallel line. He found that in this case, not only is the sum of the angles in a triangle less than 180°, but that sum depends on the area of the triangle.
  2. This result seems counterintuitive. The sum of the angles in a triangle should not depend on the size of the triangle.

III. Real progress on the parallel postulate came from Gauss.

- **A.** Gauss did not set out to prove the parallel postulate. He was interested in geometry on various surfaces, such as the surface of a sphere or a cone. He explored the idea of distance on these surfaces—curves that trace the shortest distance between two points—as well as areas of parts of these surfaces.

B. Gauss realized that the key to finding distances was to go back to Leibniz's idea of the differential—the infinitesimal change—and build up the distance by piecing together little straight lines and then use the techniques of calculus to define the distance.

C. On the surface of a sphere, the shortest distance between two points is given by what's called the *great circle*. This is the arc defined by slicing the sphere with a plane that goes through the two points as well as the center of the sphere. The distance along the great circle is the shortest distance between the two points. On a sphere, these great circles play the role of straight lines.

D. We can build triangles on a surface using great circles.

1. For example, we can construct a triangle on the surface of the Earth with a great circle that goes from the North Pole down along a line of longitude to the equator, another great circle that follows a line of longitude to the equator, and a third great circle that follows the equator itself.
2. Each of these lines of longitude hits the equator at a right angle; thus, we have a triangle, but the sum of the angles is greater than 180°.
3. For any complete great circle on the surface of the Earth and any point not on that great circle, if we look at the great circle that goes through that point, it must intersect the first great circle.
4. In other words, there is no such thing as a pair of parallel lines on the surface of a sphere.
5. This result agrees with Saccheri's prediction that if there are no parallel lines, then the sum of the angles of any triangle must be greater than 180°.

E. Gauss was also interested in *curvature*.

1. A flat surface is said to have curvature 0. A sphere with a radius of 1 is said to have curvature equal to 1. The curvature is the reciprocal of the radius. As the radius gets smaller, the curvature gets larger. As the curvature gets closer to 0, the surface begins to flatten out.
2. Not every surface, however, is the surface of a sphere. The general way to determine curvature is to look at a part of the surface and try to find that sphere that most closely approximates the way that surface bends at that point. A surface might have different curvatures at different points.

3. A surface might also bend in different directions. Consider a saddle, for example, which curves upward from front to back but curves downward from side to side. Gauss called this *negative curvature*.
4. A good example of a surface that has negative curvature everywhere is *Enneper's surface*. Such surfaces with negative curvature are often aesthetically appealing and structurally sound for use in architecture or sculpture.

F. On the surface of a sphere we have positive curvature, and we know that the sum of the angles of any triangle on this sphere will be greater than 180°. Lambert had said that the sum of the angles would depend on the area of the triangle. Gauss found a relationship among the curvature, the area of the triangle, and the sum of the angles that is today known as *Gauss's theorem*.

1. We subdivide the triangle into little pieces. We then multiply the curvature by the area for each of these pieces and add those results. We redo this computation as the pieces become smaller and smaller.
2. The limit of this sum of curvatures multiplied by area is the *total curvature* of the triangle.
3. Gauss proved that the sum of the angles in any triangle on any surface is 180° plus the total curvature. In the case of a sphere of radius 1, the curvature is always equal to 1. Thus, the sum of the angles of any triangle on that sphere is 180° plus the area of the triangle.

IV. All the surfaces Gauss worked with existed in Euclidian space. Could space itself be non-Euclidian?

A. Two mathematicians in the early 19th century explored this question. The first was János Bolyai (1802–1860), whose father, Farkas Bolyai, had been a schoolmate of Gauss.

1. In 1832, the younger Bolyai included his work on the possibility that space might be non-Euclidian in a mathematics book published by his father.
2. When Gauss read this portion of the book, he said that the ideas coincided with his own, but he did not intend to publish them. Although his investigations of surfaces had almost certainly led him to explore non-Euclidian geometry, Gauss had not perfected his ideas and did not believe the time was right for publication.

3. After Gauss's death, his manuscripts revealed that he had made many discoveries earlier than other mathematicians, and often in more complete form. Mathematics in general would have benefited if Gauss had been more willing to share his results.

B. The other 19th-century mathematician who worked on non-Euclidian geometries was Nikolai Ivanovich Lobachevsky (1792–1856).

1. Lobachevsky was a Russian from Kazan who also had a connection to Gauss: The two shared a mathematics teacher in Martin Bartels.
2. Lobachevsky corresponded with Gauss on the subject of non-Euclidian geometry, and Gauss shared Lobachevsky's ideas with Bolyai.

C. It was not until later in the 19th century, however, that non-Euclidian geometry would be accepted and fully worked out.

V. We close this lecture with Gauss's study of elliptic functions, which will be important for later lectures.

A. Earlier, we saw that Euler thought of the sine function as a function of a variable that is the arc length of a circle of radius 1.

B. We can easily define the sine for any number from 0 to 2π, but there's no reason to stop there. Once we get beyond 2π, we simply repeat the values of the sine function up to 4π, then repeat them up to 6π, and so on. We can also continue in the negative direction. We can build up the sine function as a periodic function (a function that repeats), and the same is true of the cosine function.

C. We can do the same thing with the exponential function in the imaginary direction. The exponential function is $e^{ix} = \cos x + i \sin x$. Here, once we reach $e^{i2\pi}$, we are back to 1, and we can continue the function beyond that. As we travel up the imaginary axis, the values of the exponential function will continually repeat.

D. We have trigonometric functions that repeat as we move in the real direction and the exponential function that repeats as we move in the imaginary direction. Can we have a doubly periodic function,

that is, a function that repeats in both the real and the imaginary directions?

1. French mathematician Adrien-Marie Legendre had studied such doubly periodic functions, and Gauss became fascinated by them.
2. Today, we call these *elliptic functions* because they have a connection, although an obscure one, to the ellipse. The name derives from the use of *elliptic integrals* to find doubly periodic functions. Such functions are also sometimes referred to as *Abelian functions*, named after Niels Henrik Abel, whom we'll meet in the next lecture.
3. A doubly periodic function repeats as we move horizontally and as we move vertically. The values for this function are completely determined by what happens inside a rectangle called the *fundamental domain*.
4. If we move off the right side of the fundamental domain, we will come back in on the left side. We are moving into the next copy of the fundamental domain, and the values will be exactly the same as the values that we would get coming in from the left side.
5. One way to think about that fundamental domain is to wrap it into a cylinder. Then, as we move off one side, we come back around on the opposite side.
6. The same thing happens if we go off the top of the fundamental domain; we will come back around on the bottom.
7. The resulting shape is a doughnut, or a *torus*—the natural shape for elliptic functions. Functions defined for complex values often exist on such strange surfaces.

E. We begin to see a connection between geometry and its interesting surfaces with problems in calculus and the study of functions. In the next lecture, we'll see how the idea of transformations will play a fundamental role in the development of algebra in the 19th century.

Suggested Readings:

Gindikin, *Tales of Mathematicians and Physicists*, 263–309.

Gray, *Worlds Out of Nothing*, 79–129.

James, *Remarkable Mathematicians*, chaps. on Legendre, Germain, and Gauss.

Katz, *A History of Mathematics*, 621–37, 766–79.

Questions to Consider:

1. What *is* geometry?
2. Since the space we live in is not, strictly speaking, Euclidean, what is the role of Euclidean geometry?

Lecture Eighteen

Algebra Becomes the Science of Symmetry

Scope: This lecture explores the fundamental change in algebra in the 19^{th} century, when this branch of mathematics became a tool for the study of transformations. This change was brought on by the problem of finding the exact roots of polynomials of degrees five and higher. Two mathematicians involved in early work on this question were Niels Henrik Abel and Evariste Galois, both of whom died tragically young. Abel was the first to prove that it is not always possible to find a root for an arbitrary fifth-degree polynomial. Galois solved the problem of determining when higher-degree polynomials have roots given by closed expressions. He realized that what was needed to find a closed expression for the roots of a polynomial is that the locations of the roots on the complex plane must fall into a symmetric pattern. It was this insight that brought about the change in algebra. We close the lecture with a look at Jacob Jacobi and his work on the theta function.

Outline

I. In this lecture, we look at the fundamental change that took place in algebra in the 19^{th} century, when it became a tool for the study of transformations. The origins of this change lie in the problem of finding the exact value of the root of a polynomial, the exact value where a polynomial is equal to zero.

II. Once we know one of the roots of a polynomial, we can use that information to reduce the degree of the polynomial, which gives us a simpler polynomial to work with. Thus, the real problem is finding just one of the roots of an arbitrary polynomial.

- **A.** The quadratic equation tells us how to find the root of a quadratic polynomial. The method for finding the exact value of a root for a cubic polynomial was found by del Ferro and Tartaglia. The method for a fourth-degree, or quartic, polynomial was found by Ferrari. The natural next question is: Can we find the exact value of the root of a polynomial of degree five?

B. We need to be clear about what we mean by the "exact value."

1. For a quadratic, cubic, or quartic polynomial, we can find an expression for the root that involves rational numbers together with addition, subtraction, multiplication, division, and the taking of radicals. In other words, we might need to use square roots with a quadratic equation and cube roots with a cubic equation.
2. We will allow ourselves fifth roots, or sixth roots, or whatever kind of radical we need, but we want to express the root of the polynomial precisely and not use an approximation. We want the exact value using rational numbers, the four arithmetic operations, and the radicals.

C. In 1771, Joseph Louis Lagrange did an intensive study of what was known about polynomials of degrees two, three, and four. The paper he published led the mathematical community to suspect that there was no method for finding the exact value for all fifth-degree polynomials.

D. There are fifth-degree polynomials for which we can find the exact value of the root. For example, the exact value for the root of $x^5 - 2$ is $\sqrt[5]{2}$. But can we find a closed expression for the root of an arbitrary fifth-degree polynomial?

E. In 1798, Gauss showed how to construct a regular polygon with 17 sides using a compass and straightedge. The key to this construction was finding the exact value of a root of the polynomial:

$$256x^8 + 128x^7 - 448x^6 - 192x^5 + 240x^4 + 80x^3 - 40x^2 - 8x + 1.$$

F. Gauss also proved, in his doctoral thesis of 1799, that every polynomial always has a root, though the solution may be a complex value. This is now known as the *fundamental theorem of algebra*.

III. Norwegian mathematician Niels Henrik Abel (1802–1829) was the first to prove that it is not always possible to find the exact value of a root for an arbitrary fifth-degree polynomial.

A. Abel was born into a family of modest means, and when his father died while he was still a teenager, he took on much of the financial support for the family.

B. When Abel came up with a proof of the fact that there isn't a closed solution to every fifth-degree polynomial, his teacher applied to the Norwegian government to fund Abel's studies on the Continent.

C. Abel traveled to Berlin and eventually Paris. While there, he contracted tuberculosis and died while still a young man.

IV. Evariste Galois (1811–1832) was still a teenager when he solved the problem of determining when polynomials have exact roots.

A. Galois failed the entrance exam for the École Polytechnique in 1828, when he was only 16 years old, and failed again a year later. But in 1829, he found the fundamental reason why it is that some polynomials have exact values for the roots and some don't.

1. Galois wrote up his method of analyzing polynomials, which later would be known as *Galois theory*, and sent his first draft to the French mathematician Augustin-Louis Cauchy.
2. Cauchy, who was not particularly interested in this question at the time, lost the manuscript and never made any commentary on it.

B. In 1830, Galois was admitted to the École Normale, the university established by the First French Republic to train teachers. He was expelled just a few months later because of his political activities in favor of reestablishing a republic in France.

C. About this time, Galois wrote another draft of his results on the roots of polynomials, which he sent to Fourier. Unbeknown to Galois, Fourier was on his deathbed, and the second draft of Galois' manuscript again went unread and was lost.

D. Sophie Germain became a supporter of Galois and encouraged him to rewrite his manuscript and send it out to several other mathematicians, including Siméon-Denis Poisson and Sylvestre François de Lacroix. The copy that went to Lacroix also seems to have been lost. Poisson read his copy but couldn't make any sense of it and filed it away. Long after Galois' death, this copy of the manuscript was rediscovered.

E. Even after the restoration of the French monarchy, Galois continued to demonstrate in favor of a republic for France. He was arrested twice for what were viewed as seditious activities.

F. Shortly after Galois was released from prison the second time, he found himself challenged to a duel. Galois, only 20 years old at the time, realized on the eve of the duel that he probably would not survive. That night, he wrote a letter to a friend describing his work on finding roots of polynomials. He died the next day.

G. In the 1840s, there was a resurgence of interest in the question of finding the exact values of roots of polynomials.
   1. In 1842, Charles Hermite published a proof of the impossibility of finding the root of a general quintic polynomial.
   2. Shortly after that, the Poisson manuscript of Galois' work was discovered. Joseph Liouville published and clarified Galois' work in 1846, showing the mathematical community the depth of understanding behind the work.
   3. Other mathematicians began to work on Galois theory, and many more papers were published on this subject in the 1840s and 1850s. In 1866, Serret, in the third edition of his fundamental work on algebra, *Cours d'algebre*, included a section on Galois theory.

V. The idea behind Galois theory is closely related to transformations.

A. A polynomial of degree two has two roots. A polynomial of degree three has three roots. These roots exist in the complex plane. For a polynomial of degree three, some of the roots might be at the same point in the plane, but as long as we count the roots according to their multiplicity, we will have the same number of roots as the degree of the polynomial.

B. The key to finding an exact expression for any one of these roots is based on the symmetry of the roots. One root of the polynomial $x^5 - 2$ is $\sqrt[5]{2}$; we can find all five of those roots easily by looking in the complex plane at the circle of radius $\sqrt[5]{2}$.

C. The five roots of $x^5 - 2$ are equally spaced in the complex plane around that circle of radius $\sqrt[5]{2}$. For this particular polynomial, the roots have a high degree of symmetry. If we rotate the entire plane by 72°, that transformation maps each of the roots to another one of the roots.

D. Galois realized that the trick to finding a closed expression for the roots of a polynomial is that the locations of the roots must fall into a symmetric pattern that allows transformations that take the roots into each other. When the roots exhibit this kind of symmetry, they will have closed formulas.
   1. If we have only a few roots, we always have this symmetry. For a quadratic polynomial, either both of the roots are real, or if they are complex, one of them lies above the real axis, and the other one is the mirror image of that point across the real axis.
   2. The same thing happens with polynomials of degree three and four. There is always enough symmetry in these polynomials to guarantee that we can find exact values for the roots.
   3. In a polynomial of degree five, however, we no longer have a guarantee that there is sufficient symmetry to be able to find exact values of the roots.

E. Largely because of this discovery by Galois, in the 19th century, algebra changed from looking at algebraic expressions as we think of them in high school algebra and became the study of transformations. Today, this field is called *modern algebra* or *abstract algebra*.

VI. We close with the German mathematician Jacob Jacobi (1804–1851).

A. Although he was Jewish, Jacobi was able to attend the University of Berlin and obtain his doctorate. He was unable, however, to get an academic position at any of the major German universities. He converted to Catholicism and received his first position at the University of Königsberg. In 1844, he was appointed to a chair in mathematics at the University of Berlin. In 1851, he died of smallpox.

B. In 1829, Jacobi published a groundbreaking work called *The New Fundamental Theory of Elliptic Functions*. He was interested in the fundamental domain for elliptic functions. The fundamental domain we saw in the last lecture was a rectangular region, but there is no reason it has to be rectangular; it might be a parallelogram.
   1. We can always rotate and rescale the elliptic function so that one of the periods is 1 unit to the right. The second period, then, can be any complex number; and we can assume that it's a complex number with a positive imaginary part.
   2. Thus, we have one period that is real and a second period that is a complex imaginary number. This gives us a fundamental domain that is a parallelogram.

3. Once we know the values of the elliptic function inside this parallelogram, we can translate the parallelogram horizontally or vertically to find the values of the elliptic function anywhere.

C. If we're given an elliptic function, it's possible to figure out what the period is, that is, what the shape of the parallelogram is. Jacobi asked: Given the shape of the parallelogram, is it possible to construct an elliptic function that has that particular shape?

1. This is equivalent to the following problem: If we are given a complex number, can we construct an elliptic function with that particular period?
2. The key to finding an elliptic function with this particular period is the *theta function*.

D. The theta function has as its input the complex period and as its output one of the various parameters needed to construct the elliptic function. The theta function has an interesting symmetry.

1. The function doesn't change if we replace the variable by one more than that variable, and it essentially doesn't change if we replace the variable by its negative reciprocal. This is the modular transformation that we saw in Lecture Sixteen.
2. These theta functions of Jacobi that tell us how to construct an elliptic function with a given period are said to be invariant under the transformations of the modular group.

E. This takes us back to the portion of the complex plane that lies above the real axis with smaller and smaller copies branching off as they get closer to the real axis. This is exactly the picture represented in Escher's print *Circle Limit III*. Functions that are unchanged under modular transformations are today known as *modular functions*.

Suggested Readings:

James, *Remarkable Mathematicians*, chaps. on Abel, Jacobi, Liouville, and Galois.

Katz, *A History of Mathematics*, 662–70.

Van der Waerden, *A History of Algebra*, 76–134.

Questions to Consider:

1. What *is* algebra?

2. There are two aspects to the development of new mathematics: the discovery or invention of new ideas and techniques, and the process of clarification and explanation so that others can use these ideas. History tends to celebrate the former and ignore the latter. Why?

Lecture Nineteen

Modern Analysis—Fourier to Carleson

Scope: By 1800, calculus, also known as analysis, was well established as a powerful tool for solving practical problems, but its logical underpinnings were shaky. Several factors contributed to the need to reexamine the foundations of calculus. One of the most important was the introduction of infinite sums of trigonometric functions by Joseph Fourier in 1807. Another was the extension of calculus to the realm of complex numbers. By the 1820s, Cauchy was embarked on the task of doing for calculus what Euclid had accomplished for geometry. Abel and Dirichlet moved Cauchy's work forward. In the 1850s and '60s, Weierstrass would consolidate these insights and lay the groundwork for important new discoveries.

Outline

I. This lecture looks at developments in calculus in the early 19^{th} century.

A. As noted in an earlier lecture, calculus builds on the work of Leibniz, who explained it in terms of infinitesimals: quantities that are greater than zero yet less than any positive number. Infinitesimals are difficult to understand, even for modern scientists.

B. In 1734, Bishop George Berkeley (1685–1753) argued against the claim that science is rational, in part by pointing out that calculus rests on an irrational foundation.

C. But how can we make sense of calculus if we eliminate infinitesimals? What do we really mean by the derivative $\frac{dy}{dx}$ if it isn't a ratio of infinitesimals? What do we mean by the integral of a function multiplied by dx if dx isn't the infinitely narrow width of a rectangle?

D. In the 1700s, mathematicians worked with the idea of infinitesimals without worrying about it too much, but in the early 1800s, three events forced the mathematical community to reexamine the foundations of calculus with an eye toward eliminating infinitesimals.

1. In the 1700s, only a small number of scientists used calculus, but in the early 1800s, large numbers of young engineers were being trained at the École Polytechnique and elsewhere. These students needed to understand calculus before they could apply it.

2. Another factor in the need to understand the foundations of the derivative and the integral in calculus in the early 1800s was that some mathematicians were beginning to apply calculus to complex-valued functions.

   a. It's relatively easy to explore various aspects of the curve of a function that has real numbers as input and output.

   b. However, for a function that has complex numbers as input and output, the domain is in two-dimensional space, and the range is in two-dimensional space. In order to graph the function, we need to work in four-dimensional space with the two-dimensional plane of the input orthogonal to both of the dimensions of the two-dimensional plane of the output. What do we mean by the "integral" in this case? What do we mean by the "derivative" in this case?

3. The third and most important driver behind the need to understand the foundations of calculus was the development of Fourier series.

II. Joseph Fourier (1768–1830) was a fascinating figure.

A. As a young man, Fourier intended to become a monk. He had entered a monastery but realized, after the outbreak of the French Revolution in 1789, that life would be much more exciting in Paris. Once there, he attached himself to the scientific community and learned a great deal of science and mathematics.

B. In 1794, the École Normale was established to train future teachers of science and mathematics. Fourier enrolled as one of the school's first students. A year later, he became a professor at the École Normale, and shortly thereafter moved to a position at the École Polytechnique.

C. Fourier would eventually become a science advisor to Napoleon. On Napoleon's expedition to Egypt, he identified Egyptian antiquities to be shipped back to France. He later wrote a two-volume work describing the archaeological finds made in Egypt.

D. When Fourier returned to France, Napoleon appointed him as precept of the department of Isére, which has its capital at Grenoble. Fourier's job was to create a governmental structure for the region from the ground up. At the same time, Fourier engaged in scientific work.

III. Many scientists of the time were involved with the project of modeling the flow of heat. They were searching for equations and mathematical models that showed how heat moved through a body.

A. Fourier realized that the flow of heat was related to the modeling that Euler and d'Alembert had done in the 18th century with vibrating strings. In fact, the same kinds of differential equations that are used to model vibrating strings can be modified to model the flow of heat.

1. One of the keys to modeling a vibrating string is to use the trigonometric sine and cosine functions. These are periodic functions that describe a wave, which is what we expect to see when we look at a vibrating string.
2. Fourier was interested in how heat would spread through a thin bar with a uniform supply of heat along one edge. He needed to describe this constant heat in terms of these oscillating functions, the sine and cosine.
3. In mathematical terms, Fourier needed to write the constant function 1 as a sum of cosine functions, and he found that he could do that:

$$1=\frac{4}{\pi}\left(\cos\left(\frac{\pi x}{2}\right)-\frac{1}{3}\cos\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\cos\left(\frac{5\pi x}{2}\right)-\frac{1}{7}\cos\left(\frac{7\pi x}{2}\right)+\cdots\right).$$

4. If we take just the first cosine function, we see the oscillation, but as we add more of the cosine functions together, the oscillations become smaller and tighter, and the sum of the cosine functions begins to close in on the constant function 1.
5. If we take 100 of these terms, we see just a little wiggling around the constant function 1.
6. As we take x out to -3 or $+3$, the function jumps down to -1 and then jumps back up to $+1$.

B. Mathematicians wondered about this discontinuous function. When can a function be represented as a sum of cosines or as a sum of cosines and sines?

1. This process of taking a function and decomposing it into a sum of sines and cosines would come to be known as *Fourier analysis*. One way to think about the process is in terms of music and vibrating strings.
2. In music, we get different overtones by plucking different parts of the string. Each of the cosine functions represents one of the harmonics, one of the overtones of the base tone.
3. Fourier analysis takes a complex tone and breaks it down into a base tone and its various overtones.
4. It seems strange that this process relates to heat, but as scientists would discover throughout the 19th and 20th centuries, almost every physical quantity can be broken down using Fourier analysis.

C. Fourier set out a method for discovering what these trigonometric functions should be, but how do we know that this infinite sum of trigonometric functions approaches the function in which we're interested? This is the question of the convergence of the Fourier series, and it would be unsettled until the middle of the 20th century.

IV. Augustin-Louis Cauchy (1789–1857) was undoubtedly the greatest of the French mathematicians of the 1820s.

A. Cauchy taught at the École Polytechnique, and in 1821 he started writing his great book on calculus, the *Cours d'analyse*. The work was a disaster as a textbook, but it became the basis for Cauchy's reexamination of the foundations of calculus.

B. To understand this work, let's consider a problem of differential calculus, finding the slope of the function x^3 at $x = 1$.

1. We approximate the slope by looking at the difference in the function divided by the difference in the variable. If, for example, we look at the function x^3 at 1 and at 1.1, the change in the function is $1.1^3 - 1^3$. If we divide this by the change in x, which is $1.1 - 1$, we get 3.31.
2. As we take smaller and smaller changes in x, this ratio gets closer to 3. Leibniz had said that if we take an infinitesimal change in x, we would get a ratio that is exactly equal to 3. This is what Cauchy would not allow.
3. According to Cauchy, the ratio cannot be larger than 3 because we can always take a change in x that is small enough so that the ratio is less than this larger number. We can also force this

ratio to be larger than any number that is less than 3 just by taking the change in x to be sufficiently small.

4. This is the basis for Cauchy's view of calculus, which built on the ancient Greek method of comparing two irrational numbers.
5. This view would also become our modern basis for calculus, although today we couch it in the language of epsilons and deltas.

C. In the course of his work, Cauchy made errors and introduced confusion. One of the young mathematicians in Paris at the time, Niels Henrik Abel (1802–1829), commented on this confusion to his teacher in Norway.

1. We see an example of the kinds of problems Cauchy introduced in his "proof" that every sum of continuous functions is continuous. As Abel wrote, "It appears that this theorem suffers exceptions."
2. The Fourier series is a prime example of an exception to Cauchy's rule. It is an infinite sum of continuous functions, but it gives us a discontinuous function that jumps between +1 and −1.

D. Important work on the question of when the sum of sines and cosines converges to the original function was done by another young mathematician of the 1820s, Gustav Lejeune Dirichlet (1805–1859).

1. In 1829, Dirichlet showed that Fourier analysis is valid for any *monotonic function*, a function that is always increasing or always decreasing.
2. That idea can be extended to show that Fourier analysis is valid as long as the function changes direction a finite number of times.
3. There are functions, however, that oscillate infinitely even within a finite interval, and Dirichlet's analysis doesn't work for those functions.

E. Throughout the 19^{th} century, many mathematicians explored the question of convergence of the Fourier series for functions that oscillate infinitely often. The final theorem would not be established until 1966, by Swedish mathematician Lennart Carleson (b. 1928).

1. Carleson proved that if the square of the function can be integrated, then not only does the Fourier series exist but it also converges to the original function.
2. Carleson has done much important work in analysis throughout his career and was awarded the Abel Prize in 2006.

V. We close with a mathematician who set the stage for significant work in calculus in the 19^{th} and 20^{th} centuries, Karl Weierstrass (1815–1897).

A. Weierstrass started his career as a law student but was expelled for his involvement in pranks and duels. He ultimately attended the Münster Academy and became a *gymnasium* (high school) teacher.

B. Christoph Gudermann was an instructor at the Münster Academy. He had studied elliptic functions, also known as Abelian functions, in detail and taught a class on them to just one student, Weierstrass.

C. In 1854, Weierstrass published a groundbreaking paper called "On the Theory of Abelian Functions," in which he laid out a new and comprehensive way of looking at these functions.

D. In 1855, Weierstrass was awarded an honorary doctorate from the University of Königsberg, and shortly after that he obtained an adjunct position at the University of Berlin. He would go on to take over a chair of mathematics at the University of Berlin in 1864.

E. From the 1850s and into the 1870s, Weierstrass lectured on analysis at the University of Berlin, clarifying many of the ideas that Cauchy, Abel, and Dirichlet had wrestled with in the 1820s and 1830s.

F. Weierstrass realized the importance of understanding the complexity of the real number line. One of his students was Georg Cantor, who would go on to develop set theory and explore these complexities.

G. Weierstrass did not publish many of his results, but he was an outstanding teacher who trained the next generation of German mathematicians. A total of 12,508 mathematicians can trace their mathematical genealogy back to Weierstrass.

H. One of Weierstrass's students was Sofya (or Sonya) Kovalevskaya (1850–1891).

1. Kovalevskaya was able to go to university as an undergraduate, but she could not find a university that would

accept her for graduate work. Instead, she arranged to attend Weierstrass's classes at the University of Berlin.

2. In 1874, when Kovalevskaya completed her doctoral thesis on elliptic functions, Weierstrass managed to convince his colleagues at the University of Göttingen to award her a degree.
3. In 1889, Kovalevskaya became the first woman to hold the position of chair of mathematics in a European university when she was appointed to the University of Stockholm.

VI. In the next lecture, we will look at Gauss's most famous student, Bernhard Riemann, who also did his undergraduate work with Weierstrass. Riemann fundamentally changed the way we think about questions in complex analysis, geometry, and number theory.

Suggested Readings:

James, *Remarkable Mathematicians*, chaps. on Fourier, Cauchy, Dirichlet, Weierstrass, and Kovalevskaya.

Katz, *A History of Mathematics*, 704–29.

Questions to Consider:

1. Calculus existed and was used for almost 150 years before mathematicians were finally confronted with the need to establish its logical underpinnings. What forced this change?
2. Weierstrass published a lot of important mathematics, but his greatest influence on the development of mathematics probably was through his lectures and his doctoral students. What do you believe was the role of his own training and experience as a high school teacher?

Lecture Twenty

Riemann Sets New Directions for Analysis

Scope: The figure who dominated German mathematics in the 1850s and 1860s was Bernhard Riemann. His doctoral dissertation established new insights into complex analysis, while his oral habilitation presentation launched what would later become known as *Riemannian geometry*, a branch of geometry that would provide key insights for Einstein. Riemann is also known for his work with the prime number theorem, establishing the framework that would enable two French mathematicians to prove this result in the 1890s. At the turn of the 20th century, modern geometry and algebra began to merge thanks to the work of Arthur Cayley, Felix Klein, David Hilbert, and others.

Outline

I. Bernhard Riemann (1826–1866) dominated German mathematics in the late 1850s and well into the 1860s.

A. Born in Hanover, Riemann attended the University of Göttingen, initially with the intention of studying theology. There he met Gauss, who suggested he transfer to the University of Berlin.

B. Riemann completed his undergraduate work at Berlin, then returned to Göttingen for his doctoral work and his habilitation (a more advanced thesis that involves an oral defense).

C. At the time that Riemann was a graduate student in Göttingen, Richard Dedekind (1831–1916) was also studying with Gauss. Dedekind later explored the nature of the real number line and was responsible for the *Dedekind cut*, one way to define irrational numbers.

D. In 1859, Riemann was appointed to the chair of mathematics at the University of Göttingen. In the same year, he published an important paper on the prime number theorem. Riemann was set for an auspicious career, but in 1862 he contracted tuberculosis.

E. Riemann died in 1866, leaving much of his most important work unpublished. Dedekind later prepared his manuscripts for publication.

II. In his doctoral dissertation, Riemann explored the problems of calculus applied to complex numbers (complex analysis).

A. We begin with complex-valued functions of a complex variable. As an example, there is the function that maps a complex number z to the exponential e^z.

1. This function maps the real number line to a ray that starts at the origin (but does not include the origin) and travels infinitely far off to the right.
2. We now move up and consider the image of the horizontal line whose imaginary part is $\left(\frac{\pi}{4}\right)i$. If we take e raised to a point on this line, the line becomes a ray that goes out from the origin at an angle of 45°.
3. If we move up again to the horizontal line whose imaginary part is $\left(\frac{\pi}{2}\right)i$, the line is mapped to a ray that goes vertically upward. If we take horizontal lines at $\frac{\pi}{4}i, \frac{\pi}{2}i, \frac{3\pi}{4}i$, and so on, we will get rays going out from the origin at 45° angles from each other.
4. This function maps the vertical imaginary axis to the circle of radius 1 that is centered at the origin.
   a. A vertical line with real part equal to 1 maps to a circle with radius equal to e^1. A vertical line with real part equal to 2 maps to a circle with radius equal to e^2.
   b. A vertical line with real part equal to −1 maps to a circle with radius equal to e^{-1}, a radius less than 1.
   c. The images of horizontal lines are rays going out from the origin. The images of vertical lines are concentric circles with their center at the origin.
5. In the original domain, the horizontal and vertical lines meet at right angles. In the image, the rays and circles also meet at right angles. This is not coincidental.

B. We can take any two intersecting curves in the domain and consider the angle between the tangent lines at the point of intersection.

1. If we look at the images of these curves using the exponential function, the image curves will meet at exactly the same angle.
2. A function that preserves angles in this way is called a *conformal mapping*. Most familiar functions of a complex variable, such as polynomials, trigonometric functions, and rational functions (ratios of polynomials) are conformal mappings.

C. In his dissertation, Riemann reversed the problem of preservation of angles. He asked: Given a region in the domain and another region in a range, is there always a complex-valued function that provides a conformal mapping from the first region to the second? The answer that Riemann found is "yes," provided the regions are not too strange.

III. The prime number theorem, the result Riemann studied in 1859, provides an approximation to the number of primes below a given number.

A. The primes are the numbers that cannot be written as a product of two integers larger than 1, numbers such as 2, 3, 5, 7, and 11.

B. The prime-counting function is a step function. Every time we reach a new prime, we increase the value by 1.

C. In 1792, Gauss realized that the right function for estimating the number of primes less than x is the *logarithmic integral*.

1. One function that can be used to estimate the number of primes less than x is x divided by the natural logarithm of x ($x/\log x$).
2. A more accurate estimate, however, is given by the logarithmic integral, li(x), which is the area underneath the reciprocal of the natural logarithm between 2 and x ($\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$).

D. If we go up to $x = 100{,}000$, we see that the prime-counting function stays well above $x/\log x$ and just a little bit below li(x).

E. Even up to very large values of x, the prime-counting function seems to stay below the logarithmic integral, but John Edensor Littlewood proved that the prime-counting function is not always less than the logarithmic integral.

1. In fact, the prime-counting function fluctuates above and below the logarithmic integral. As we go out toward infinity, the prime-counting function and the logarithmic integral interchange places infinitely often.

2. The first value at which the prime-counting function is larger than the logarithmic integral is known as the *Skewes number*, which has only recently been calculated to be about 1.39×10^{316}.

F. In order to study the prime-counting function, Riemann looked at the zeta function.

1. The *zeta function*, $\zeta(z)$, is the sum of the reciprocals of the integers raised to a power that is an arbitrary complex number: $\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$. If the real part of z is larger than 1, then this infinite summation converges.
2. According to Riemann, the fact that this function is defined when the real part of z is larger than 1 implies that it can be defined for all complex values except $z = 1$.
3. Riemann realized that understanding this function would enable a proof that the number of primes less than or equal to x is well approximated by the logarithmic integral of x. All that was needed to prove this was to show that the zeta function does not take on the value 0 for any complex number of the form $z = 1 + iy$ (real part of z equal to 1).
4. In the 1890s, two French mathematicians, Jacques Hadamard and Charles Jean de la Vallée Poussin, finally proved the prime number theorem, following the outline laid out by Riemann.

IV. Riemann went even farther, looking at the size of the error term. How close does the prime-counting function stay to the logarithmic integral? The key to this question was to find the other places where the zeta function could equal 0, the zeros of the zeta function.

A. Riemann showed that the zeta function has a kind of symmetry about the vertical line with real part $\frac{1}{2}$. Thus, if we know the values on one side of that line, we can take the mirror image of those values to find the value of the zeta function on the other side.

B. We know the values of the zeta function when the real part of z is larger than 1 and can use the symmetry to find the zeros when the real part of z is less than 0.

C. The difficulty comes for complex numbers where the real part lies between 0 and 1. This is known as the *critical strip*.

1. Riemann showed that if all of the zeros of the zeta function that lie within this critical strip lie along the single vertical line with real part $\frac{1}{2}$, then the difference between the prime counting and the logarithmic integral is never more than about $\sqrt{x}$, much smaller than the size of the logarithmic integral.
2. This statement about the location of the zeros became known as the *Riemann hypothesis*: All of the zeros of the zeta function that lie inside the critical strip lie on a single vertical line with real part equal to $\frac{1}{2}$.

D. We now know that most of the zeros of the zeta function in the critical strip lie on this vertical line, and in 2004, Xavier Gourdon showed that the Riemann hypothesis holds for the first 2.4 trillion zeros, but the Riemann hypothesis is still unproven and remains one of the most important open problems in mathematics today.

V. We now turn to some of the work Riemann did with geometry.

A. Gauss had used differential calculus to define distances and areas on a given surface. Riemann inverted the problem, starting with an arbitrary definition of a distance and looking at the geometry it implies. This would be the root of what is today called *Riemannian geometry*.

B. Riemann's work in this area would be a key insight for Einstein. In general relativity, Einstein came up with a particular definition of distance that was consistent with his understanding of the physical world. He then went back to Riemannian geometry to see what his definition of distance implied about the nature of the geometric space in which he was working.

C. Another important contributor to geometry at this time was Arthur Cayley (1821–1895). He showed that projective geometries can be explained in terms of differential geometries.

D. Felix Klein (1849–1925) is known for the *Klein bottle*, a three-dimensional extension of the idea of the Möbius strip.

1. The Klein bottle is a three-dimensional object that is given a twist through the fourth dimension so that there is no distinction between the inside and the outside of the bottle.
2. The Möbius strip is a two-dimensional object that is given a half twist through the third dimension so that it becomes an object that has only one side.

3. Klein also sought to understand *invariants*, that is, transformations that leave geometries unchanged. With his *Erlangen program*, Klein set out to characterize all possible geometries and show how they could be explained in terms of invariants.

E. As we've seen, algebra is the study of transformations, and in the late 19th and early 20th centuries, geometry and algebra began to merge into an area of mathematics known as *algebraic geometry*. This branch of mathematics looks at the algebra that sits behind geometric spaces.

F. Klein went on to become the chair of mathematics at Göttingen and was followed in that position by David Hilbert (1862–1943).

1. Göttingen became the world's greatest center for mathematics in the early 20th century, but the scholarly community there was destroyed by the exodus of Jewish mathematicians following the rise of Hitler.
2. At the International Congress of Mathematicians in 1900, Hilbert described 23 problems that would motivate much of mathematics in the 20th century.

Suggested Readings:

Gindikin, *Tales of Mathematicians and Physicists*, 311–22.

Gray, *Worlds Out of Nothing*, 187–232, 251–59.

James, *Remarkable Mathematicians*, chaps. on Riemann and Hadamard.

Katz, *A History of Mathematics*, 726–29, 737–46.

Laugwitz, *Bernhard Riemann, 1826–1866*.

Markushevich, "Analytic Function Theory."

Questions to Consider:

1. What are the advantages of considering functions defined over all complex numbers rather than just the real numbers?
2. Why was Riemann's approach to geometry so powerful?

Lecture Twenty-One
Sylvester and Ramanujan—Different Worlds

Scope: This lecture introduces us to James Joseph Sylvester and Srinivasa Ramanujan, two fascinating mathematicians who worked with the question of counting partitions, that is, the number of ways an integer can be written as a sum of positive integers. Because he was Jewish, Sylvester had difficulty in both getting admitted to university and finding a teaching position. Later in life, he accepted a position at The Johns Hopkins University, where he set up the graduate program in mathematics, began to build a mathematical research community in the United States, and founded the first journal devoted to mathematics in this country. Ramanujan was born in India and had no formal training in mathematics. His correspondence with G. H. Hardy, however, revealed such deep insights that Hardy brought Ramanujan to England to continue his mathematical work. Ramanujan returned to India during the last year of his life and worked feverishly on the theta functions that Jacobi had studied. Since the 1970s, his papers have sparked exciting investigations into theta, modular, and elliptic functions.

Outline

I. In this lecture, we meet James Joseph Sylvester and Srinivasa Ramanujan.

A. Both of these mathematicians were interested in *partitions*, the number of ways an integer can be written as a sum of positive integers. For example, 5 can be written as a sum of positive integers in seven different ways: 5, $4+1$, $3+2$, $3+1+1$, and so on.

B. This may seem like a rather uninteresting problem, but it has deep repercussions, and extensions of this idea play out in modern physics today.

II. James Joseph Sylvester (1814–1897) was Jewish and, like Jacob Jacobi, ran into difficulties both getting admitted to university and finding a university position.

A. Sylvester attended Cambridge, but he was never allowed to officially matriculate there because he did not belong to the Church of England.

B. Sylvester took a position teaching natural philosophy (physics) at University College in London and eventually received a master's degree from Trinity College in Dublin.

C. In 1841, Sylvester accepted a job as a professor of mathematics at the University of Virginia, which had been founded in 1819. The students treated the faculty terribly, and Sylvester left after only a few months.

D. He stayed in the United States for another year and a half, trying to find an academic position. He was unsuccessful and eventually returned to England, where he found work as an actuary and also studied law.

E. Finally, in 1855, Sylvester secured a position teaching mathematics at the Royal Military Academy in Woolwich. He came into conflict with the administration there over the issue of whether the faculty at universities should be given time to conduct research.

F. In 1870, when he reached the age of 55, Sylvester was forced into retirement, although he was by no means ready to retire. Fortunately, in 1877, he was hired by The Johns Hopkins University in Baltimore, Maryland, to set up a graduate program in mathematics.

G. Sylvester now had the time and resources to focus on his own research and begin building a research mathematical community in the United States. At the same time, he founded the first journal devoted exclusively to mathematics in the United States, the *American Journal of Mathematics*.

1. Sylvester encouraged the students in his classes to pursue interesting questions; he then gathered their work and had it published in the journal.
2. One such paper was "A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion," in which we can read the voices of both Sylvester and several of his students.

H. Sylvester was highly successful in training research mathematicians who would go on to work at other great universities in the United States and build their graduate programs.

I. In 1883, Sylvester was finally recognized in Britain as the great mathematician that he was. He was appointed to the position of Savilian Professor at Oxford, the same chair in mathematics that John Wallis had held in the 17th century.

III. Srinivasa Ramanujan (1887–1920) was born in Tamil Nadu in southern India and grew up in Kumbakonam. He was a member of the highest caste in India, the Brahmin.

A. Despite their high caste, Ramanujan's family was not wealthy. His father worked as a clerk in a store, and in order to supplement their income, the family took in boarders, students from the nearby government college.

B. Ramanujan was extremely talented in mathematics from an early age. He began to study the mathematics textbooks that the students living at his house would bring home.

C. One of the books that had a profound influence on Ramanujan was G. S. Carr's *A Synopsis of Elementary Results in Pure Mathematics*, written to help prepare students for the Tripos exam at Cambridge. Ramanujan was fascinated by this dense listing of results and formulas and began to explore some of the ideas they represented.

D. Among the topics Ramanujan studied were the Bernoulli numbers, Stirling's theorem, and the Euler-Maclaurin formula. In 1904, he entered college but failed his examinations in every subject except mathematics and was forced to leave. Two years later, he attended another college with the same result.

E. For the next several years, Ramanujan traveled around southern India, staying with friends and family members and carrying his mathematics notebooks with him. Eventually, he realized that he needed to find a source of income and to connect with the wider mathematical community.

F. Ramanujan met Ramachandra Rao, a prominent civil servant and secretary of the local section of the Indian Mathematical Society. Rao became Ramanujan's patron and introduced him to other mathematicians.

G. With the encouragement of these Indian mathematicians, Ramanujan wrote to scholars in England to see how his work connected to what others were doing. One of the people he contacted was G. H. Hardy, probably the greatest mathematician in England at the time.

H. As Hardy read Ramanujan's letter and looked over some of his results, he realized they had been written by someone with a deep understanding of mathematics. Hardy shared the letter with J. E. Littlewood, his colleague at Cambridge, and the two wrote back to Ramanujan, asking him to explain some of his methods.

I. Ramanujan did not have formal training in mathematics, but he understood its patterns and structure and used that understanding to arrive at his results. Although he made some errors, he also had deep insights.

J. Hardy and Littlewood wanted to bring Ramanujan to England to fill in some of the gaps in his education and to work with him directly. Ramanujan was reluctant to leave India, however, because in doing so he would lose his caste. After a period of prayer and fasting, Ramanujan decided that it was his fate to go to England; he traveled there in the summer of 1914.

K. Ramanujan had a tough time maintaining his dietary restrictions in England and was unaccustomed to the cold climate. His years in England were difficult, but he did brilliant mathematics, and his collaboration with Hardy was important. Among other insights, the two mathematicians discovered a formula for finding the number of partitions of very large numbers.

L. In 1918, Ramanujan was elected as a fellow of the Royal Society. A number of stories attest to his mathematical prowess, and it has been said that every number was a personal friend to Ramanujan.

M. In 1919, Ramanujan returned to India hoping to restore his health, but he died within a year, at the age of 32. In the last year of his life, however, he worked feverishly on mathematics, particularly on mock theta functions.

1. Recall that Jacobi had used theta functions in the 1820s to find elliptic functions with a particular period. The theta functions are characterized by certain transformation invariance.
2. Ramanujan looked at functions that have many of the same properties as theta functions but lack their transformation invariance. He showed that many of the properties thought to stem from the transformation invariance actually came from other properties that applied to a much larger class of functions.
3. In the early 1960s, George Andrews, a graduate student at the University of Pennsylvania, studied these mock theta functions for his doctoral dissertation. In 1976, Andrews traveled to England, where he discovered some of the work Ramanujan had done on mock theta functions during the last year of his life.
4. The work that Ramanujan had done between 1919 and 1920 had direct applications to investigations being conducted in the 1970s in the area of statistical mechanics. In the decades since 1976, exciting mathematics has been done in the study of theta functions, modular functions, and elliptic functions building on Ramanujan's results.

Suggested Readings:

Gindikin, *Tales of Mathematicians and Physicists*, 337–47.

James, *Remarkable Mathematicians*, chaps. on Jacobi, Hardy, and Ramanujan.

Kanigel, *The Man Who Knew Infinity*.

Katz, *A History of Mathematics*, 687–97.

Questions to Consider:

1. Few topics in mathematics are as amenable to clever manipulation as elliptic and modular functions. Despite the fact that most mathematicians working in this field do it for the pleasure of playing with these fascinating objects, the results have often proven to be useful in modeling our physical world. Is this something that should be expected?
2. It can be very difficult today for anyone to be taken seriously as a research mathematician if they have not gone through traditional graduate training in mathematics. Have we lost the opportunity to identify a future Weierstrass or Ramanujan?

Lecture Twenty-Two
Fermat's Last Theorem—The Final Triumph

Scope: This lecture follows the search for a proof of Fermat's last theorem. We investigate the proof of the theorem for $n = 5$ by the unlikely team of Germain, Legendre, and Dirichlet. Lamé saw the connection between Fermat's last theorem and algebraic integers, generalizations of Gauss's complex integers, and announced a general proof in 1847. Kummer found the fatal flaw in Lamé's work, revealing unexpected complexity in algebraic integers. In 1877, Richard Dedekind would provide the definitive understanding of these generalized integers. Later mathematicians found a connection between the numerators of Bernoulli numbers and Fermat's last theorem, but the final proof was derived from the study of elliptic and modular functions.

Outline

I. In this lecture, we look at the proof of Fermat's last theorem. It's interesting to note that mathematicians really don't have much use for the result itself, but the process of trying to find the proof opened up new areas of mathematics with important applications.

II. Recall that Fermat had referred to the theorem in a marginal note written in his copy of *Arithmetica* by Diophantus.

A. Fermat had been looking at Pythagorean triples, integers that satisfy the equation $x^2 + y^2 = z^2$. Examples of Pythagorean triples include 3, 4, 5; 5, 12, 13; 8, 15, 17; and so on. We know that there are infinitely many of these triples.

B. Fermat asked the question: Can three positive integers be found that satisfy the equation $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$, or $x^n + y^n = z^n$ for some $n > 2$? According to Fermat, he had worked out a proof that the answer was "no," but he couldn't fit the proof in the margin of *Arithmetica*. He did prove that there is no solution for the exponent 4, but he almost certainly did not know how to prove the other cases.

C. In the 18th century, Euler proved that there is no solution for the exponent 3. At the beginning of the 1800s, the next case to be proved was for the exponent 5.

III. Among the mathematicians involved in the proof that there is no solution to the equation $x^5 + y^5 = z^5$ were Sophie Germain (1776–1831), Gustav Lejeune Dirichlet (1805–1859), and Adrien-Marie Legendre (1752–1833).

A. Germain showed that if there are three integers that satisfy $x^5 + y^5 = z^5$, then one of the three integers must be divisible by 5.

B. Gustav Lejeune Dirichlet, then 20 years old, showed that if a solution exists for the case of the exponent 5, then the integer that is divisible by 5 must be odd.

C. In 1825, Adrien-Marie Legendre, then 72 years old, tackled the problem and showed that the integer that is divisible by 5 must be even. Because there are no integers that are both even and odd, the equation can't be solved for the case of the exponent 5.

D. Dirichlet returned to the problem in 1832 and showed that no solution exists when the exponent is 14. He was able to skip some of the exponents because, for example, if no solution exists for the exponent 3, then there cannot be a solution for the exponent 6.

E. The case of the exponent 7 was solved by Gabriel Lamé (1795–1870) in 1839. In 1847, Lamé announced that he could extend his work to prove Fermat's last theorem in general.

1. To understand Lamé's key idea, we return to the Pythagorean triples. One of the keys to finding all the possible triples of integers that satisfy the equation $x^2 + y^2 = z^2$ is to factor the polynomial on the left side of the equation ($x^2 + y^2$).
2. We can factor $x^2 - y^2$ into $(x-y)(x+y)$, but $x^2 + y^2$ doesn't seem to factor. We can, however, factor this polynomial if we introduce complex integers (integers that have a real part and an imaginary part that is also an integer, e.g., $2+3i$). The result is: $x^2 + y^2 = (x+iy)(x-iy)$. We can use this factorization to discover all the Pythagorean triples.

3. As we begin to expand our idea of *integer* to these *complex*, or *Gaussian, integers*, factorization changes. Some of the prime numbers remain prime, but some do not.
4. With Gaussian integers, 2 can be factored: $2=(1+i)(1-i)$; 5 can also be factored: $5=(1+2i)(1-2i)$. The numbers 2 and 5 are no longer prime numbers in the Gaussian integers.
5. The proof that Fermat's last theorem is true for the exponent 3 uses the same sort of idea. The polynomial x^3+y^3 can be factored as $(x+y)(x^2-xy+y^2)$, but it can't be completely factored into linear factors.
6. If we extend our idea of what constitutes an integer and introduce $\omega=e^{2\pi i/3}$, a cube root of 1, then we can factor x^3+y^3 completely into linear factors: $x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$. This factorization is key to the proof of the impossibility of finding a solution to $x^3+y^3=z^3$.

F. In his work of 1847, Lamé was trying to do this factorization in general. It is possible to factor x^n+y^n into linear factors (that is, into n products of simple binomials) by introducing integers that use the n^{th} roots of unity.

G. Lamé presented his result to the Academy of Science in Paris in 1847, engendering much discussion. Liouville, whom we met earlier in connection with Galois, pointed out a basic flaw in Lamé's approach to proving Fermat's last theorem.

1. In the ordinary integers, there is only one way of writing an integer as a product of primes. For example, $6=2\times3$ or 3×2, but we can't multiply two completely different primes to get 6.
2. Lamé had assumed that this would continue to be true for the extended integers involved in the n^{th} roots of 1. Liouville, however, said that there might be cases in which the factorization was not unique. In other words, there might be more than one way of choosing the primes that are multiplied together to get one of these extended integers.

3. On March 15, 1847, Pierre Wantzel announced that he had a proof that this factorization would in fact be unique, and thus Lamé's approach should be correct.
4. Liouville suspected that Wantzel's proof was incorrect. This was confirmed by Ernst Kummer in Berlin.
5. As Kummer would later show, if we look at the prime exponent 37 and take extended integers that involve the 37^{th} root of 1, the factorization is not unique. The proof that Lamé thought he had for Fermat's last theorem fell apart.

H. Kummer continued to work on this problem of factorization and to study extended integers. One of the things he showed was that if the factorization is unique for a given prime, then Fermat's last theorem is correct for that prime exponent.

1. The extended integers would come to be called *algebraic integers*. Kummer explored their factorization and the ways in which they differ from ordinary integers.
2. This work on algebraic integers came to fruition in 1877 with the publication of Richard Dedekind's *The Theory of Algebraic Integers*.

IV. Eventually, a connection was discovered between the numerators of the Bernoulli numbers and Fermat's last theorem.

A. If p does not divide any of the numerators of the Bernoulli numbers up to the $p-3^{rd}$, then the factorization is unique for the algebraic integers that involve the p^{th} root of 1. In other words, Fermat's last theorem is correct for that exponent p.

B. Eventually, this result was extended. What if p divides one of those numerators, but only one of them, and p^2 does not divide any of the numerators? In that case, too, Fermat's last theorem can be proven. What if p^2 divides one of the numerators, but p^3 can't be divided into the product of those numerators? In that case, Fermat's last theorem is also true.

C. Mathematicians began building up cases for which Fermat's last theorem is true, that it is impossible to find a triple of integers that satisfies the equation $x^p+y^p=z^p$. By 1976, it was known that Fermat's last theorem is true for all exponents below 100,000, but a new approach was needed to prove it for *all* exponents greater than 2.

D. A breakthrough came in 1983. German mathematician Gerd Faltings (b. 1954) showed that for each exponent greater than or equal to 3, there are, at most, finitely many solutions. Recall that the Pythagorean triples had infinitely many solutions.

E. In 1985, two mathematicians, Andrew Granville and Roger Heath-Brown, working independently, showed that Fermat's last theorem must be true for infinitely many exponents, and in some precise mathematical sense, it must be true for most of the exponents.

V. The final approach to Fermat's last theorem came from the study of elliptic and modular functions.

A. Weierstrass had published an important work on Abelian functions in which he found an efficient way of describing elliptic functions in terms of two parameters that are today referred to as g_2 and g_3.

B. The values g_2 and g_3 uniquely determine the elliptic function (now known as a "$\wp$-function"). Weierstrass found the differential equation that is satisfied by $\wp$: $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

C. As succeeding mathematicians studied these elliptic functions, they found that it was useful to look at the surface that corresponds to this equation: $y^2 = 4x^3 - g_2x - g_3$.

1. Normally, such an equation would be graphed as a curve on the plane, but in this case, the input and output are complex numbers. The space of all solutions to this equation, then, is a two-dimensional surface that sits in four-dimensional space.
2. This two-dimensional surface in four-dimensional space is known as an *elliptic curve*.
3. Further, we want our surface to include the point at infinity; thus, we must take the projection of this curve in much the same way that we looked at projective space earlier, when we replaced each of the lines through the origin with a single point.
4. The elliptic curve is a strange surface that is difficult to visualize, but it is useful in answering many questions in mathematics and analysis.

VI. During the 1950s, a number of mathematicians looked at elliptic curves that come from modular functions, which are generalizations of theta functions.

A. Modular functions are invariant under the kinds of transformations that we saw represented in Escher's print *Circle Limit III*.

1. Each modular function gives rise to an elliptic curve, and mathematicians of the 1950s began to suspect that every elliptic curve also corresponds to one of these modular functions.
2. If true, that statement would connect two very different areas of mathematics—the geometry of elliptic curves and the analysis of modular functions.

B. Two Japanese mathematicians working together, Yutaka Taniyama and Goro Shimura, and a French mathematician working independently, André Weil, made the conjecture that every elliptic curve is modular.

1. Thinking back to analytic geometry, we know that every algebraic equation gives rise to a geometric curve. Not every geometric curve, however, comes from an algebraic expression.
2. What if it were true that every algebraic curve, or every curve that had sufficiently nice geometric properties, corresponded to an algebraic equation? That kind of correspondence would be useful.
3. Taniyama, Shimura, and Weil conjectured exactly that kind of statement for elliptic curves—that every elliptic curve corresponds to a modular function.

C. In 1980, Jean-Pierre Serre and Gerhard Frey showed that if Fermat's last theorem is false, then three integers exist that can be used to construct an elliptic curve that might not correspond to a modular function.

D. In 1986, Ken Ribet showed that they were right, that this curve could not correspond to a modular function. If Fermat's last theorem is false, then not every elliptic curve is modular. On the other hand, if we know that every elliptic curve is modular, then Fermat's last theorem must be true.

E. In 1993, Andrew Wiles at Princeton announced that he had proved the Taniyama-Shimura-Weil conjecture for a class of elliptic curves that included the Serre-Frey curve built on a counterexample to Fermat's last theorem. Thus, Wiles announced that Fermat's last theorem is true because every elliptic curve of the appropriate kind is modular. In 1995, this proof of Fermat's last theorem was finally published.

F. Initially, Fermat's last theorem was proven as a special case of the Taniyama-Shimura-Weil conjecture. The full Taniyama-Shimura-Weil conjecture would be proven in 1999 by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, thus closing one of the most important stories in the history of mathematics.

Suggested Readings:

Edwards, *Fermat's Last Theorem.*

James, *Remarkable Mathematicians*, chaps. on Legendre, Germain, Dirichlet, Kummer, and Dedekind.

Kline, *Mathematical Thought from Ancient to Modern Times*, chap. 34.

Van der Poorten, *Notes on Fermat's Last Theorem.*

Questions to Consider:

1. Fermat thought that he could prove the result that came to be known as "Fermat's last theorem." Do you think that he really had a complete proof?
2. What are other examples in mathematics or science of a problem for which the solution was of less importance than the tools that were developed to arrive at that solution?

Lecture Twenty-Three
Mathematics—The Ultimate Physical Reality

Scope: The Newtonian paradigm for science is to seek understanding of the world by building a mathematical model. Beginning with electricity and magnetism in the mid-19th century, this has proved to be a fruitful way of exploring physical reality because the mathematics itself suggests phenomena that had not been expected. We shall see this in Maxwell's equations, which led to the discovery of radio waves, and in Einstein's special and general theories of relativity, which grew out of the Lorentz transformations. We shall explore how mathematics provides the ultimate explanation of subatomic particles, and we shall indicate the role of mathematics in some of the modern theories of the universe, including string theory.

Outline

I. Our intuition serves us well when we're looking at things on the scale of a meter, a centimeter, or a millimeter. But as soon as we try to extend our understanding to the cosmic scale—to speeds that approach the speed of light or to the subatomic scale—our intuition fails us. In these realms that are outside of our intuition, we need to rely on mathematics.

   A. In this lecture, beginning with Maxwell's equations and continuing through general relativity, quantum mechanics, and modern string theory, we'll see how mathematics enables us to work with physical realities for which our experience fails us.

   B. We've already seen some of the ways in which mathematics suggests excess content. For example, Fourier found a connection between the transfer of heat and the mathematics of a vibrating string. It's precisely this kind of unexpected excess content, which stems from applying mathematics to physical reality, that makes mathematics so important in understanding the fundamental nature of reality.

II. We begin with the story of electricity and magnetism.

   A. Initially, the study of electricity was the study of static electricity, but in the late 1700s, the voltaic pile—what today we would call a *battery*—was invented, and it became possible to generate a current.

B. People were also interested in magnetism, and in 1820, Danish scientist Hans Christian Ørsted noted a connection between electricity and magnetism. He discovered that closing the circuit of an electrical apparatus caused the needle on a nearby compass to move. Clearly, the electric current created some kind of magnetic field.

C. When Ørsted shared his discovery with the rest of the scientific world, others began to explore the relationship between electricity and magnetism. Later in 1820, two French scientists, Jean-Baptiste Biot and Félix Savart, came up with the first important mathematical model of the connection between electricity and magnetism, now known as the *Biot-Savart equation*.

D. In 1831, Michael Faraday found that not only does an electric current create a magnetic field, but a magnetic field can be used to create an electric current. This idea is behind our modern generation of electricity, which is accomplished by *dynamos*: rotating magnets that induce a current.

E. Out of Faraday's work, an equation was developed to explain how the magnetic field interacts with electrical current. This equation depends on the three spatial dimensions and the dimension of time, because it is the change in the magnetic field that is critical in generating the electric current.

F. Another important piece of our understanding of electricity and magnetism came from Gauss, who was interested in the question of electrostatic potential.

1. Newton had explained gravitational attraction in terms of forces. Gravity exerts a force, which produces an acceleration or change in velocity.
2. By the late 1700s, astronomers realized that they should consider potential energy instead of forces. Think of a book lying on the floor. If we raise the book up into the air, it now has the potential to develop energy as it falls; we say that the book that we hold above the floor has *potential energy*. That potential energy can then be translated into kinetic energy as the book falls.
3. The force of gravity tends to move an object from an area of high potential energy to one of low potential energy. The advantage of working with potential energy is that it is a single number, not a vector as force is.
4. Once we know the energy level of an object at each point in space, we know that the object will be pulled in the direction of greatest decrease in its potential energy. If we know the energy levels, we can determine the forces.
5. Gauss developed the mathematical model that explains electrostatic potential energy.

G. In the 1860s, British scientist James Clerk Maxwell decided to apply what Gauss had done for electrostatics to electrodynamics. He wanted to find an electrodynamic potential that could be used to simplify the various equations that related electricity and magnetism. In 1864, he published his results in a groundbreaking paper, "A Dynamical Theory of the Electromagnetic Field."

1. Maxwell's four equations are often stated separately, but they can be united if we think about the electromagnetic potential.
2. One of the curious things Maxwell discovered is that, just like Fourier's heat flow, the electromagnetic potential is related to the problem of a vibrating string.
3. In the case of a string, the vibrations happen only in one direction, up and down. Euler and others in the 1700s extended this to the idea of a drumhead, where the drumhead is a two-dimensional surface that vibrates in the third dimension.
4. Maxwell's electromagnetic potential is actually a vibrating three-dimensional object, although the direction in which it vibrates is not clear.
5. As we know, if we pluck a string or disturb a pool of water, the disturbance spreads out. Maxwell realized that exactly the same sort of thing should happen with electromagnetic potential.
6. Mathematically, he established that his electromagnetic potential should spread out in all three dimensions. Experimentally, he discovered that the electromagnetic potential should travel at the speed of light, thus suggesting that there is a fundamental connection between light and the electromagnetic potential.

III. At this point, electromagnetic potential was only a mathematical construct. Scientists who were studying electricity and magnetism at this time wondered if there was any reality to the electromagnetic potential.

A. In 1887, German scientist Heinrich Rudolf Hertz found a way to detect electromagnetic potential and, thus, prove its existence in reality.

B. This discovery suggested the possibility of communication over long distances without any physical connection. If it is possible to detect changes in electromagnetic potential, then it should be possible to manipulate an electric current to cause changes in the electromagnetic potential; this disturbance, which should travel at the speed of light, might then be used to send a message.

C. In 1895, two scientists, Alexander S. Popov and Guglielmo Marconi, working independently, figured out a means of carrying out this communication. What they invented was the radio.

1. A radio picks up changes in the electromagnetic potential; the waves of electromagnetic potential are what we today refer to as *radio waves*.
2. No one would have suspected the existence of radio waves had it not been for Maxwell's purely mathematical work.

D. A number of scientists then began studying electromagnetic potential in more detail. Two of them, Irish scientist G. F. Fitzgerald and Dutch scientist Hendrik Lorentz, realized that there is a kind of transformation of space-time that leaves the electromagnetic potential unchanged.

1. This transformation, which came to be known as the *Lorentz transformation*, in some way twists space and time. It doesn't change the basic equation of electromagnetic potential, which suggests that electromagnetic phenomena are invariant under this transformation.
2. One of the consequences of this invariance is that, for an electromagnetic phenomenon moving close to the speed of light, distances contract by a factor of $\sqrt{1-\frac{v^2}{c^2}}$, where v is the speed of the object and c is the speed of light.

E. In 1905, Albert Einstein suggested that this contraction, this invariance under Lorentz transformations, doesn't just happen for electromagnetic phenomena, but for everything. This is the basis for special relativity.

F. Einstein realized that time itself would contract, as well. This gives rise to an apparent paradox: A person traveling in a rocket ship at a speed near the speed of light will experience less change in time than a person who remains on Earth. In fact, this phenomenon has been verified with extremely accurate clocks sent into orbit at high velocities.

IV. Einstein then applied the idea of invariance under Lorentz transformations to the geometry of space itself.

A. Earlier in the course, we saw Riemann's idea that the basis for understanding geometry is to begin with differentials that define distances. We can define the distances any way we want, and that in turn determines the geometry of the space we're working in.

B. Felix Klein further developed this idea, showing that we don't have to start with distances. We can start with transformations that leave the geometry unchanged. Later mathematicians built on these ideas, developing the tools needed to learn what these transformations tell us about the nature of this geometry.

C. Einstein drew on Riemannian geometry to explore the nature of a universe that is invariant under the Lorentz transformations. Amazing predictions followed, for example, that a massive gravitational body should distort the shape of space and that light itself would be bent in the presence of a massive body. These predictions would later be verified.

V. We now turn to quantum mechanics.

A. We usually think of electrons as particles that whiz around the nucleus of an atom, but as scientists realized early in the 20th century, electrons are much more complex than that.

B. In 1922, Louis de Broglie showed that an electron can be thought of as both a particle and a wave. This work related to an earlier controversy concerning light, which Einstein had showed was both a particle and a wave.

C. In 1926, Erwin Schrödinger found the differential equation—the equation involving derivatives—that explained this wave-particle behavior.

D. In the same year, Max Born realized that the kind of function that satisfies Schrödinger's equation is actually a probability distribution, a way of describing where something is likely to be.

1. An electron should be thought of as a probability distribution; it doesn't have an actual position, only a likelihood that it will be within a certain range of positions.
2. Our intuition breaks down at the subatomic level. We have to rely on mathematics to try to understand what happens.

E. Werner Heisenberg worked with the idea that all subatomic particles are probability distributions, which led to his uncertainty principle: This probability function has the strange property that if we try to pin down how fast the particle is moving, we will necessarily lose certainty about its position. If we try to pin down its position, we will lose certainty about its velocity.

F. There is a basic problem with Schrödinger's equation explaining subatomic particles: This equation is not invariant under the Lorentz transformation.
1. Einstein saw clearly that all of the universe must be invariant under this transformation of four-dimensional space-time, but Schrödinger's equation does not fit that reality.
2. Schrödinger's equation must be a special case of something more general that is, in fact, invariant under the Lorentz transformations.
3. Combining Lorentz invariance with the Schrödinger equation would become the goal of scientists in the 20th and 21st centuries. This raises the question: What kinds of transformations preserve both Lorentz invariance and all the symmetries that are inherent in Schrödinger's equation?

VI. One of the mathematical models proposed to answer this question is today known as *string theory*.

A. Many people are amazed that string theory, depending on which version is under discussion, is about objects that exist in 10-, 11-, or even 26-dimensional space.
1. These extra dimensions come from the need to preserve many symmetries and transformations, which can't be done in 4 dimensions.
2. Physicists have a hard time explaining why we don't experience these other dimensions, but what matters is that if we want to preserve all the symmetries we know about, we need this many dimensions.

B. String theory is interesting because it is built on modular functions. It asserts that the basic building block of all matter in the universe

is the modular function. This strange function, with the invariance represented by Escher's *Circle Limit III*, somehow encodes some of the properties of the mathematics that sits at the foundation of the universe.

C. There is a good deal of controversy about string theory. As we saw in Lecture One, Wigner said that physics sees patterns in the world and looks for mathematics that seems to fit those patterns. That's exactly what is happening in string theory.
1. We see the patterns—the Lorentz invariance and the transformations preserved by Schrödinger's equations—and we're trying to find the mathematics that fits those patterns. String theory, with its use of modular functions, seems to fit.
2. But as Wigner said, there's no reason to believe that mathematics pulled from one area is relevant to another.

D. There is currently no way to test string theory and no way to know whether this particular mathematics is the right mathematics to understand subatomic particles. A prominent skeptic of string theory, Freeman Dyson, however, has admitted that the theory is so beautiful that it must be useful for something, whether or not it proves to be the foundation of the subatomic world.

Suggested Readings:

Dyson, "Missed Opportunities."

Gindikin, *Tales of Mathematicians and Physicists*, 369–82.

Penrose, *The Road to Reality*.

Questions to Consider:

1. Both light and subatomic particles participate in wave-particle duality: If we study them as if they were waves, they exhibit wave-like behavior; if we study them as if they were particles, they exhibit particle-like behavior. What does this tell us about the reality of light and the reality of subatomic particles?
2. At the scales of the very large and the very small, our imagination fails us because the realities of the world that we experience at our own scale fail to apply. Why is mathematics so effective at providing the models that work at these scales, and is there any hope for a more intuitive understanding of the nature of reality at these scales?

Lecture Twenty-Four

Problems and Prospects for the 21st Century

Scope: Mathematics today has more interesting and important problems than ever in its history. New connections are being discovered, and new fields of study are being born. This last lecture explores new sources for questions in mathematics, including information technology and the human genome. We will also look at the millennium problems put forth by the Clay Institute, particularly the P versus NP problem, the Riemann hypothesis, and the recently proved Poincaré conjecture. Mathematics continues to be an exciting and dynamic field, expanding our understanding of the world by combining knowledge in new ways.

Outline

I. In this last lecture, we'll see some of the excitement and richness that we can find in mathematics today.

- **A.** In the first lecture of this course, we talked about the fact that mathematics comes from the world around us through the exploration of patterns.
- **B.** We've seen that mathematics grew out of questions in commerce and civil administration, navigation and surveying, astronomy, the physical world, and the subatomic world. We've also seen how mathematics has grown through art and our understanding of symmetries and transformations.
- **C.** Today, we have new sources of patterns that suggest problems for mathematicians to work on.

II. One of these sources of patterns is information technology.

- **A.** An example of the mathematics that has grown from information technology can be found in the public key cryptosystem developed by R. L. Rivest, A. Shamir, and L. Adelman (RSA) at MIT in the 1970s.
- **B.** This system encrypts information so that it can be sent securely over the Internet, but the key for the encryption must be created in such a way that seeing how the information is encoded does not help with decoding.

- **C.** The idea that sits behind the RSA public key cryptosystem is the basic mathematical fact that it's easy to multiply two very large primes but difficult to find the two primes from their product. This procedure has given rise to much interesting mathematics that explores properties of the integers.
- **D.** The Internet itself can be thought of as a graph with billions of nodes. In fact, as it continues to develop, it will explode into quintillions of nodes. How can we deal with all of this information? Mathematicians now are doing research into the deep structure of such systems.
  1. The theory group at Microsoft, led by Jennifer Tour Chayes and her husband, Christian Borgs, has been modeling the Internet and looking at problems of social networking and the spread of worms and viruses.
  2. Their work pulls in some of the ideas we've looked at from geometry, algebra, and analysis—and the interaction of these three strands of mathematics.

III. Another area that is giving rise to new and exciting problems in mathematics is biology, especially genomics.

- **A.** We can now write out the human genome as billions of letters (As, Cs, Gs, and Ts) that make up the menu for constructing a given individual. The problem now is how to interpret this mass of information.
- **B.** The genome contains the genes—the templates—for constructing proteins that are used in the functioning of the body. The problem for biologists is to figure out which pieces of this string of letters do what.
- **C.** A given gene is not just a single sequence of letters; it may have a number of sequences. How do these different sequences combine? If one sequence is part of a gene, where is the rest of it? How are the sequences related? Mathematicians are now looking at these questions of genomics for plants, animals, and humans and applying techniques from algebra, geometry, analysis, and statistics.
- **D.** One mathematician who has done important work in this area is Eric Lander of Harvard University. He has a background in both mathematics and biology and was involved in the Human Genome

Project in its early days. In fact, Lander is the first author listed on the final classification of the human genome.

E. Bernd Sturmfels, working at the University of California at Berkeley, has found inventive ways to combine ideas from statistics, algebra, and geometry to identify genes, their functions, and how they change over time.

F. The human genome isn't just a list of instructions for how to create a person; it is also a record of all the different experiments that nature has carried out on the creatures that eventually have led to humans. Once we decode this record, we will have an incredible wealth of information that will greatly advance our understanding of biology.

G. Sturmfels and his team have identified a particular gene that is 42 letters long and is common to all vertebrates. Somehow, this gene seems to be essential to being a vertebrate animal; the sequence has been dubbed "the meaning of life."

IV. Not only does mathematics look at questions from the world around us, but it also continues to generate many of its own questions.

A. In the year 2000, the Clay Institute in Cambridge, Massachusetts, put out the millennium problems, the greatest unsolved problems in mathematics, in order to spur the work of mathematicians. The institute has offered a $1 million prize for the solution to each of these problems.

B. One of the problems is called the *Birch and Swinnerton-Dyer conjecture*, a conjecture about elliptic curves.

C. Another open problem is the *Hodge conjecture*, which involves algebraic geometry.

D. The third problem deals with the existence and smoothness of solutions to the *Navier-Stokes equations*. These are the mathematical equations that describe fluid dynamics. Turbulence plays a role in this problem.

E. The fourth problem is the *Poincaré conjecture*, and the fifth is a problem from computational complexity related to developing algorithms to work with computers. It's called the *P versus NP problem*.

1. This problem looks at algorithms for which the running time is a polynomial in the input variable.

2. Polynomial time is faster than exponential time. In polynomial time, doubling the input will multiply the running time by some finite power of 2. In exponential time, doubling the input squares the running time.

3. The question is, if it's possible to verify that a solution is correct in polynomial time, is it also possible to find a solution in polynomial time?

F. Another problem is proving the *Riemann hypothesis* that bounds the error when the prime-counting function is approximated by the logarithmic integral. This problem comes down to finding the places within the critical strip where the Riemann zeta function is zero and determining whether or not they all lie on the single vertical line with real part equal to $\frac{1}{2}$.

G. The seventh problem is from mathematical physics—finding *quantum Yang-Mills theory*.

V. The Poincaré conjecture is the one problem in this group that has been solved.

A. The problem was solved by a Russian mathematician, Grigori Perelman (b. 1966), in 2002. In 2006, Perelman was offered the Fields Medal as an acknowledgment of his work in mathematics, but he declined it, saying that people should not do mathematics for the sake of awards or remuneration.

B. One of the problems that interested Poincaré was the basic stability of our solar system.

1. As we know, the planets travel around the Sun in essentially elliptical orbits, but their motions are complicated by the forces of the Sun and the other planets pulling on them.

2. Poincaré and others wondered whether these perturbations would ever build up to the point where one of the planets would leave its elliptical orbit and spiral into the Sun. Newton had been unable to establish the fact that the orbits were basically stable.

C. Poincaré noted that as a planet, such as the Earth, orbits the Sun, we cannot predict with certainty where it will be at each pass because we can never pin down its exact location. Any small change in that location will magnify as the planet continues to move around the Sun.

D. At the same time, however, we can establish a window through which we know the Earth will always pass on its journey around the Sun. Even though we can't say exactly where the Earth will be, we know that it will stay within this small window, and that's enough to assert that the orbit is stable.

E. This idea was the beginning of chaos theory.

1. Many people know of chaos theory as the idea that a butterfly flapping its wings in China will create enough uncertainty in the air currents to cause a hurricane over Florida. We can never know the state of the atmosphere precisely, and in fact, small changes in one place can magnify and result in dramatic changes elsewhere.
2. The idea behind chaos theory is to find the stability that sits behind this unpredictability, exactly the same kind of stability that Poincaré found. Even systems that seem to be chaotic may have some deep underlying structure.
3. Chaos theory is being used, for example, to study the underlying patterns in heart arrhythmias.

F. In developing his understanding of celestial mechanics, Poincaré also launched the study of topology.

1. As we saw earlier, the elliptic function exists on a torus, a doughnut. What's important about the torus is that it has periodicity as we move both vertically and horizontally, but it's not the same kind of periodicity that a sphere has.
2. If we travel vertically or horizontally around a sphere, we repeat, but not in the same way that we would repeat on a torus. One question that topology looks at is: What is the fundamental difference between a torus and a sphere?
3. Think of putting a rubber band around a sphere or putting it around a doughnut in such a way that the rubber band passes through the center of the doughnut. We could easily remove the rubber band from the sphere, but we could not remove it from the doughnut without cutting either the rubber band or the doughnut. That's the fundamental topological difference between a doughnut and a sphere.

G. The Poincaré conjecture looked at the surface of a sphere in four-dimensional space. The surface of a sphere in three-dimensional space is two-dimensional; thus, the surface of a sphere in four-dimensional space is a three-dimensional object.

1. One of the ways to characterize a sphere in four-dimensional space is to use the same rubber band. It's possible to remove a rubber band, or any closed curve, from a three-dimensional sphere in four-dimensional space. Topologically, we say that if we have a closed curve, we can always shrink that curve down to a single point, where it disappears.
2. Poincaré asked: Are there any other objects in four-dimensional space that are not topologically equivalent to a sphere and that have this property that every closed curve can be shrunk down to a point? (*Topologically equivalent* refers to a kind of rubber geometry, some way of stretching and bending it so that this other object could be turned into a sphere.)
3. The Poincaré conjecture is that in four dimensions, this rubber band property only applies to objects that are equivalent to a sphere. Perelman proved this conjecture by developing deep geometric and topological tools that will be applicable to many other questions.

VI. Throughout this course, we have painted a picture of mathematics as looking at how patterns overlap and seeing how we can extrapolate from the combination of different patterns and different areas of mathematics.

A. We've brought together algebra and geometry, algebra and the idea of symmetries and transformations, and analysis or calculus and geometry. People from outside mathematics often focus on mathematics as problems, but for mathematicians, the subject is all about seeing how these patterns interrelate.

B. One of the great mathematicians in the world today is Sir Michael Atiyah (b. 1929). Atiyah has held the Savilian Chair in Mathematics at Oxford; he was also the first director of the Isaac Newton Institute for the Mathematical Sciences at Cambridge, and he has won both a Fields Medal and the Abel Prize.

C. In the 1980s, Atiyah was asked how he selected a problem for study. He responded:

> I don't think that's the way I work at all … I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring things up. Things emerge, and I follow them up. Or I see something which

connects up with something else I know about, and I try to put them together, and things develop.

That is the study of mathematics.

Suggested Readings:

Carlson et al., *The Millennium Prize Problems*.

Cartwright, "Mathematics and Thinking Mathematically."

Gindikin, *Tales of Mathematicians and Physicists*, 323–35.

James, *Remarkable Mathematicians*, chap. on Poincaré.

Minio, "An Interview with Michael Atiyah."

Nasar and Gruber, "Manifold Destiny."

O'Shea, *The Poincaré Conjecture*.

Questions to Consider:

1. Is it beneficial to offer large financial incentives for the proof (or disproof) of a mathematical conjecture?
2. How does the true nature of mathematics differ from its popular perception? If these are not in agreement, what can be done to correct this?

Timeline

B.C.

2030–1640	Egyptian Middle Kingdom.
2000–1600	Old Babylonian period.
800–200	Composition of the Indian *Sulbasutras*.
c. 624–c. 545	Life of Thales of Miletus.
c. 563–483	Life of Siddhartha Gautama, the Buddha.
551–479	Life of Confucius.
c. 520	Pythagoras founds his school in Samos.
388	Plato founds the Academy in Athens.
323	Alexander the Great dies; Ptolemy I rules Egypt; Seleucus I rules Persia and Mesopotamia.
c. 300	Museion at Alexandria is founded, Euclid flourishes.
3rd century	Period of the Warring States in China.
212	Rome conquers Syracuse; death of Archimedes.
208	Founding of Han dynasty in China.
c. 200	Apollonius of Perga writes the *Conics*.
127–126	Hipparchus makes his astronomical observations in Rhodes.
30	Rome annexes Hellenistic Egypt.

A.D.

1st century	Buddhism enters China; Kushan Empire extends into northern India.
early 2nd century	Ptolemy writes the *Almagest*.

mid-3rd century	Diophantus writes *Arithmetica*.
late 3rd century	Liu Hui writes *Sea Island Computational Canon*.
c. 320	Founding of Gupta Empire in India.
337	Roman Emperor Constantine I baptized as a Christian.
391	Theodosius I orders the destruction of all pagan temples in the Roman Empire; probable end of the Museion in Alexandria.
415	Death of Hypatia of Alexandria.
late 5th century	Zu Chongzhi discovers $\frac{355}{113}$ as approximation to π.
618	Founding of Tang dynasty in China.
622	Muhammad flees from Mecca to Medina: beginning of Islamic calendar.
644–648	Li Chunfeng collects and revises existing Chinese mathematical treatises into the *Ten Computational Canons*.
mid-7th century	Brahmagupta leads the astronomical observatory at Ujjain.
750	Abbasid caliphate founded in Baghdad.
786–809	Harun al-Rashid rules in Baghdad.
825	Al-Kwarizmi writes his treatise on *al-jabr* and *al-muqabala*.
c. 1000	Al-Haytham begins his work as scientist and engineer in Cairo.
mid-11th century	Jia Xian produces first definitively known example of "Pascal's triangle."
1088	Founding of the University of Bologna.

1149	Al-Samawal writes *The Brilliant in Algebra*.
12th century	Bhaskara Acharya leads the observatory at Ujjain.
1202	Leonardo of Pisa (Fibonacci) writes the *Liber abaci*.
1235	Ujjain conquered by the Delhi caliphate and destroyed.
1258	Hulagu Khan sacks Baghdad.
1260–1294	Rule of Kublai Khan in China.
1303	Zhu Shijie writes *Trustworthy Mirror of the Four Unknowns*.
c. 1450	Gutenberg invents moveable-type printing.
1453	Constantinople conquered by Ottoman Turks.
1492	Ferdinand and Isabella conquer Granada; Columbus lands in America.
c. 1505	Del Ferro discovers method for finding roots of arbitrary cubic polynomial.
1517	Luther nails his theses to church door in Wittenberg.
1525	Albrecht Dürer publishes his book on geometric constructions.
1543	Nicolaus Copernicus publishes *De revolutionibus*.
1545	Gerolamo Cardano publishes *Ars magna*.
1581	Dutch Republic wins freedom from Spain.
1585	Simon Stevin publishes *La Thiende*, advocating the use of decimal fractions.

1601 .... Tycho Brahe dies in Prague; his assistant, Johannes Kepler, inherits his astronomical data.

1607 .... Jamestown is founded in Virginia.

1610 .... Galileo publishes *The Starry Messenger*.

1614 .... Napier publishes *Description of the Wonderful Canon of Logarithms*.

1620 .... Plymouth Colony is established in Massachusetts.

1637 .... Fermat and Descartes publish descriptions of analytic geometry.

1642 .... Death of Galileo; birth of Newton.

1644 .... Descartes publishes *Principles of Philosophy*.

1653–1658 .... Cromwell rules as Lord Protector.

1655 .... Wallis publishes *Arithmetic of Infinities*.

1660 .... Royal Society of London is founded.

1663 .... Barrow becomes first to hold the Lucasian Chair in Mathematics at Cambridge.

1684 .... Leibniz publishes his first paper on calculus.

1687 .... Newton publishes *Mathematical Principles of Natural Philosophy*.

1700 .... Prussian Academy of Sciences is founded.

1713 .... Jacob Bernoulli's posthumous *The Art of Conjecture* is published by his brother, Johann.

1727 .... Euler joins the newly founded St. Petersburg Academy of Sciences.

1744 .... Frederick the Great reestablishes the Royal Academy of Sciences in Berlin.

1748 .... Euler publishes *Introduction to Analysis of the Infinite*.

1789 .... Storming of Bastille, start of French Revolution.

1794 .... Founding of the École Normale.

1799–1825 .... Laplace publishes *Treatise on Celestial Mechanics*.

1801 .... Gauss publishes *Investigations of Arithmetic*.

1807 .... Fourier submits his thesis on *The Propagation of Heat in Solid Bodies*.

1815 .... Defeat of Napolean at Waterloo; reinstitution of the Bourbon monarchy.

1821 .... Cauchy publishes *Cours d'analyse*.

1824 .... Abel proves impossibility of solving the general quintic.

1830 .... Charles X flees France; accession of Louis-Philippe.

1832 .... Galois is killed in duel.

1848 .... End of French monarchy, beginning of rule of Napoleon III.

1848–1870 .... Independence and unification of Italy.

1863–1871 .... Unification of Germany.

1864 .... Riemann delivers his lecture "On the hypotheses that lie at the foundations of geometry."

1865 .... Maxwell publishes *A Dynamical Theory of the Electromagnetic Field*.

1878 .... Sylvester founds *American Journal of Mathematics*; Cayley publishes *The Theory of Groups*.

1887 .... Hertz detects electromagnetic potential.

1889 .... Kovalevskaya becomes first woman to hold a professorship in mathematics at a European university.

1892–1899 .... Poincaré publishes *Lectures on Celestial Mechanics*.

1896 .... Hadamard and de la Vallée Poussin independently prove the prime number theorem.

1900 .... Hilbert announces his 23 problems.

1914 .... Ramanujan travels to England to work with Hardy.

1914–1918 .... World War I.

1915–1916 .... Einstein publishes the theory of general relativity.

1922 .... De Broglie introduces the wave-particle duality of electrons.

1936 .... Ahlfors and Douglas are first recipients of the Fields Medal.

1939–1945 .... World War II.

1946 .... ENIAC computer.

1957 .... Sputnik.

c. 1970 .... Early development of string theory.

1984 .... First Macintosh computer.

1991 .... Debut of the World Wide Web.

1995 .... Wiles publishes proof of Fermat's last theorem.

2000 .... Clay Mathematics Institute announces the seven Millenium Prize Problems.

2002 .... Perelman announces proof of the Poincaré conjecture.

2003 .... Serre becomes first recipient of Abel Prize; completion of the Human Genome Project.

Glossary

algebra: The field of mathematics that deals with expressions in unspecified quantities. In its purest sense, it seeks to find the value of an unknown quantity by manipulating a balance of two different expressions in that unknown quantity. In the 19th century, the meaning of the term expanded to cover generalized number systems and the study of transformations and symmetries.

amicable numbers: A pair of numbers, such as 220 and 284, for which the sum of the proper divisors of one equals the other.

analysis: As a field of mathematics, the advanced study of calculus that appeared in the 19th century.

analytic geometry: The area of mathematics that combines algebra and geometry by representing algebraic expressions as geometric curves plotted on a Cartesian plane.

calculus: A field of mathematics with its origins in the problems of calculating slopes of tangent lines and rates of change (differential calculus) as well as areas, volumes, and other quantities that are limits of sums of products (integral calculus).

chord: The straight-line segment connecting any two points on a circle.

combinatorics: The mathematics of counting arguments.

complex number: Any sum of a real and an imaginary number. Complex numbers represent points in a plane.

continued fraction: A representation of a number as a sequence of integers obtained by specifying the integer part of the number and then iteratively finding the integer part of the reciprocal of what remains.

decimal system: A place-value system based on powers of 10.

degree: $\frac{1}{360}$ of the circumference of a circle.

digit: Any of the 10 symbols 0 through 9.

Diophantine equation: Any equation for which possible solutions are restricted to integers.

elliptic function: A doubly-periodic complex-valued function of a complex variable.

elliptic integral: Any of a family of integrals among which is the integral for determining the length of the arc of an ellipse.

Fermat's last theorem: The statement made by Fermat that for any exponent $n > 2$, there is no triple of positive integers x, y, z for which $x^n + y^n = z^n$.

Fourier series: The representation of a function by an infinite series of trigonometric functions.

fundamental theorem of algebra: The statement that every polynomial with real coefficients has at least one root (which might be complex).

fundamental theorem of calculus: The statement of the equivalence of two ways of thinking of integration: as the inverse process of differentiation and as a limit of sums of products.

geometry: The mathematical abstraction of spatial relationships.

harmonic series: The sum of the reciprocals of the positive integers.

imaginary number: The square root of a negative number.

infinite series: An infinite summation.

irrational number: A number that cannot be represented as a ratio of two integers.

isochrone: A curve with the property that a ball placed anywhere along it will take the same amount of time to reach the bottom.

logarithm: An exponent. Specifically, given a number larger than one, called the *base*, the logarithm of any positive number is the power of the base that will yield that number.

method of exhaustion: A means of finding areas and volumes by breaking the region up into ever-finer pieces.

minute (′): $\frac{1}{60}$ of a degree.

modular function: A function that exhibits invariance under a set of transformations of the independent variable that includes translations and the taking of the negative of the reciprocal.

number theory: The study of the integers, especially the pursuit of integer solutions to mathematical problems.

Pascal's triangle: A triangular arrangement of the coefficients of the expansions of powers of the binomial $1 + x$.

perfect number: A number, such as 6 or 28, that is equal to the sum of its proper divisors.

pi (π): The ratio of the circumference of a circle to its diameter.

place-value system: A method for recording numbers that assigns different values to the digits depending on their position. Thus, the 2 in 27 represents two 10s, or 20.

Platonic solids: The five solids whose faces are identical regular polygons.

polynomial: A mathematical expression involving a sum of powers of a variable in which each term may be multiplied by a number, such as $x^2 - 3x + 6$.

Pythagorean theorem: The statement that in any right triangle, the sum of the areas of the squares whose sides are adjacent to the right angle is equal to the area of the square whose side is opposite the right angle.

Pythagorean triple: Three positive integers that form the sides of a right triangle.

rational number: A number that can be represented as a ratio of two integers.

Riemann hypothesis: The statement that the nonreal zeroes of the zeta function, a function used to study the distribution of prime numbers, all have real parts equal to $\frac{1}{2}$.

root: A root of a polynomial is a value at which the polynomial is zero.

second ($''$): $\frac{1}{60}$ of a minute.

sexagesimal system: A place-value system based on powers of 60.

sine: A half-chord.

square root: Given a number that represents an area, the square root of that number is the length of the side of the square with that area.

statics: The physics of counterbalancing forces.

Sulbasutras: Appendices to the Vedas that include detailed mathematical descriptions of altar construction.

topology: A mathematical field within geometry that studies properties that are left unchanged by small changes in how distance is measured. Informally known as *rubber sheet geometry*.

trigonometry: The mathematics built on the study of chords.

variable: An unspecified quantity that can be assigned more than one value.

Vedas: Hindu mythological texts composed over the period 2500–600 B.C.

Biographical Notes

Abel, Niels Henrik (1802–1829). Norwegian mathematician who was first to prove the impossibility of a general solution to polynomial equations of degree five.

Al-Haytham (c. 965–1040). Mathematician, scientist, and engineer born in Basra; worked in Baghdad and Cairo. Noted for his work in optics and his methods for finding volumes of solids of revolution.

Al-Kwarizmi (c. 790–840). Baghdad mathematician whose *Condensed Book on the Calculation of Restoring and Comparing* is considered the first book of algebra.

Al-Samawal (c. 1130–c. 1180). Jewish doctor and scientist from Baghdad who made important advances in algebra.

Apollonius of Perga (c. 260–c. 190 B.C.). The inventor of the astronomical system of epicycles and the author of the *Conics*, which established the mathematical properties of parabolas, ellipses, and hyperbolas.

Archimedes of Syracuse (287–212 B.C.). The greatest of the Greek scientists. Among his many accomplishments was the development of the method for finding areas and volumes.

Aryabhata (476–550). Indian astronomer who worked at Kusumapura, near modern Patna.

Bernoulli, Jakob (1654–1705). Swiss scientist and author of *The Art of Conjecture*, the founding work in the theory of probability.

Bernoulli, Johann (1667–1748). Swiss scientist who worked with Leibniz, helped establish calculus, and taught Leonhard Euler.

Bhaskara Acharya (1114–1185). Last of the great astronomers at Ujjain, India. Discovered general methods for finding quadratic approximations and for solving the problem today known as Pell's equation.

Bombelli, Rafael (1526–1572). Important algebraist who introduced and explained the workings of complex numbers.

Brahmagupta (598–c. 665). Head of the astronomical observatory at Ujjain, India. Made significant contributions to trigonometry and number theory.

Cantor, Georg (1845–1918). Founder of modern set theory.

Cardano, Gerolamo (1501–1576). Preeminent algebraist of the 16th century and author of *Ars magna* (*The Great Art*).

Cauchy, Augustin-Louis (1789–1857). French mathematician; considered the founder of analysis.

Cayley, Arthur (1821–1895). British mathematician who began the unification of non-Euclidean and projective geometries and helped to establish modern algebra.

Cohen, Paul (1934–2007). The American mathematician who finally established the status of the axiom of choice and the continuum hypothesis.

Diophantus of Alexandria (c. 200–284). The first to introduce algebraic notation; author of *Arithmetica*, the book that would inspire much future work in number theory.

Euclid of Alexandria (c. 325–c. 265 B.C.). Hellenistic scholar who established the mathematical community at the Museion of Alexandria and who consolidated all Greek knowledge of mathematics.

Eudoxus of Cnidus (c. 395/390–c. 342/337 B.C.). Greek mathematician who is credited with the discovery of the method of exhaustion for finding areas.

Euler, Leonhard (1707–1783). Swiss mathematician who worked in St. Petersburg and Berlin. One of the most prolific mathematicians and probably the most influential.

Fermat, Pierre (1601–1665). Councilor to the parliament in Toulouse, inventor of analytic geometry, and one of the important contributors to the development of calculus and number theory.

Fontana, Niccolò (aka **Tartaglia**; 1499–1557). Independent discoverer of the technique for finding the exact value of a root of any cubic equation.

Fourier, Joseph (1768–1830). French mathematician, scientist, and bureaucrat.

Galilei, Galileo (1564–1642). Italian mathematician and scientist noted for his attempts to explain the physics of motion.

Galois, Evariste (1811–1832). French mathematician; first person to solve the general problem of when a polynomial equation can be solved exactly.

Gauss, Carl Friedrich (1777–1855). German astronomer and mathematician who spent most of his career at Göttingen and whose contributions span geometry, number theory, and analysis.

Germain, Sophie (1776–1831). French mathematician who made important contributions to the understanding of Fermat's last theorem.

Gödel, Kurt (1906–1978). Austrian logician who proved the incompleteness theorem.

Hilbert, David (1862–1943). German mathematician who made important contributions in algebra, geometry, and analysis.

Hipparchus of Rhodes (190–120 B.C.). Astronomer and father of trigonometry.

Huygens, Christiaan (1629–1695). Dutch scientist who invented the pendulum clock, discovered a Moon of Saturn, and served as mentor to both Leibniz and Johann Bernoulli.

Hypatia of Alexandria (c. 370–415). The first woman known to have made important contributions to mathematics, she wrote commentaries on several of the classic texts.

Jia Xian (fl. mid-11th century). Chinese mathematician; first person known to have recorded Pascal's triangle.

Klein, Felix (1849–1925). Head of the mathematics faculty at Göttingen and leader in the development of geometry.

Kovalevskaya, Sofya (aka **Sonya**; 1850–1891). Russian mathematician who worked in analysis and taught at the University of Stockholm.

Lebesgue, Henri (1875–1941). French mathematician known for his radically different approach to integration.

Leibniz, Gottfried Wilhelm (1646–1716). Librarian to the Duke of Hanover; shares with Newton the claim to be one of the founders of calculus.

Leonardo of Pisa (aka **Fibonacci**; c. 1170–1240). Merchant from Pisa who learned Islamic mathematics and wrote several important books in which he shared his knowledge.

Liu Hui (fl. late 3rd century A.D.). Earliest known Chinese mathematician; author of the *Sea Island Computational Canon*.

Napier, John (1550–1617). Scottish nobleman and theologian; inventor of the logarithm.

Newton, Isaac (1642–1727). Lucasian Professor of Mathematics at Cambridge and author of *Mathematical Principles of Natural Philosophy*.

Noether, Emmy (1882–1935). German algebraist.

Poincaré, Henri (1854–1912). French mathematician noted for his work in analysis and number theory and his work on the stability of complex systems.

Ptolemy of Alexandria (c. 100–c. 170). Astronomer and author of *Mathematiki Syntaxis*, which would come to be known as *Almagest*, the book that would shape astronomy until the 16th century.

Pythagoras of Samos (c. 580–c. 500 B.C.). Greek mystic who established a philosophical school at Croton, in what is now Italy, based on the premise that "all is number."

Ramanujan, Srinivasa (1887–1920). Self-taught Indian mathematician who worked on elliptic and modular functions.

Riemann, Bernhard (1826–1866). German mathematician who made seminal contributions to geometry, analysis, and number theory.

Stevin, Simon (1548–1620). Belgian scientist known for his contributions to algebra and statics and largely responsible for the European adoption of decimal fractions.

Sylvester, James Joseph (1814–1897). British algebraist; first professor of mathematics at The Johns Hopkins University and a founder of the American mathematical community.

Wallis, John (1616–1703). Savilian Professor of Geometry at Oxford and discoverer of many of the fundamental insights of calculus.

Weierstrass, Carl (1815–1897). Leading analyst of the mid-19th century; taught at the University of Berlin.

Wiles, Andrew (b. 1953). Princeton number theorist who proved Fermat's last theorem in 1995.

Zhu Shijie (c. 1260–1320). Chinese mathematician; author of *Siyuan yujian* (*Trustworthy Mirror of the Four Unknowns*).

Zu Chongzhi (fl. late 5th century A.D.). Chinese mathematician who discovered $\frac{355}{113}$ as an extremely accurate approximation to π.

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Bashmakova, Isabella, and Galina Smirnova. *Dolciani Mathematical Expositions*. Vol. 23, *The Beginnings and Evolution of Algebra*. Translated by Abe Shenitzer. Washington, DC: Mathematical Association of America, 2000. A comprehensive history from the geometric algebra of Babylon and Greece through the developments of the 19th century.

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Datta, B., and A. N. Singh. "Hindu Trigonometry." *Indian Journal of History of Science* 18 (1983): 39–108. An account of the trigonometric results that were known to Indian astronomers from A.D. 300 to 1500.

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———. *The Mechanization of the World Picture: Pythagoras to Newton*. Translated by C. Dikshoorn. Princeton, NJ: Princeton University Press, 1986 (1950). Originally published in Dutch, this source is now slightly outdated but is still the most thorough and erudite description of the development of our understanding of the physical world up to Newton's *Principia*.

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Gray, Jeremy. *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*. London: Springer-Verlag, 2007. An undergraduate textbook and an excellent overview of this rich thread in mathematics.

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James, Ioan. *Remarkable Mathematicians: From Euler to von Neumann*. Cambridge: Cambridge University Press, 2002. Brief biographies of 60 of the greatest mathematicians of the 18th, 19th, and first half of the 20th centuries.

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Kline, Morris. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972. Now somewhat dated and occasionally flawed, but a great sweeping account of the history of mathematics that still provides inspiring insights for those curious about the nature of the discipline.

Laugwitz, Detlef. *Bernhard Riemann, 1826–1866: Turning Points in the Conception of Mathematics*. Translated by Abe Shenitzer. Boston, MA: Birkhäuser Verlag, 1999. A wonderfully thorough account of Riemann and his mathematics. Some of it gets technical, but most of the book can be read with little knowledge of advanced mathematics.

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O'Shea, Donal. *The Poincaré Conjecture: In Search of the Shape of the Universe*. New York: Walker & Company, 2007. An engaging account of the Poincaré conjecture and its solution, written by a mathematician who truly knows this field.

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Notes